A TREATISE ON

BESSEL FUNCTIONS

AND

THEIR APPLICATIONS TO PHYSICS.

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"And as for the *Mixt Mathematikes* I may onely make this prediction, that there cannot faile to bee more kindes of them, as Nature growes furder disclosed."

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PREFACE.

THIS book has been written in view of the great and growing importance of the Bessel functions in almost every branch of mathematical physics; and its principal object is to supply in a convenient form so much of the theory of the functions as is necessary for their practical application, and to illustrate their use by a selection of physical problems, worked out in some detail.

Some readers may be inclined to think that the earlier chapters contain a needless amount of tedious analysis; but it must be remembered that the properties of the Bessel functions are not without an interest of their own on purely mathematical grounds, and that they afford excellent illustrations of the more recent theory of differential equations, and of the theory of a complex variable. And even from the purely physical point of view it is impossible to say that an analytical formula is useless for practical purposes; it may be so *now*, but experience has repeatedly shown that the most abstract analysis may unexpectedly prove to be of the highest importance in mathematical physics. As a matter of fact it will be found that little, if any, of the analytical theory included in the present work has failed to be of some use or other in the later chapters; and we are so far from thinking that anything superfluous has been inserted, that we could almost wish that space would have allowed of a more extended treatment, especially in the chapters on the complex theory and on definite integrals.

With regard to that part of the book which deals with physical applications, our aim has been to avoid, on the one hand, waste of

time and space in the discussion of trivialities, and, on the other, any pretension of writing an elaborate physical treatise. We have endeavoured to choose problems of real importance which naturally require the use of the Bessel functions, and to treat them in considerable detail, so as to bring out clearly the direct physical significance of the analysis employed. One result of this course has been that the chapter on diffraction is proportionately rather long; but we hope that this section may attract more general attention in this country to the valuable and interesting results contained in Lommel's memoirs, from which the substance of that chapter is mainly derived.

It is with much pleasure that we acknowledge the help and encouragement we have received while composing this treatise. We are indebted to Lord Kelvin and Professor J. J. Thomson for permission to make free use of their researches on fluid motion and electrical oscillations respectively; to Professor A. Lodge for copies of the British Association tables from which our tables IV., V., VI., have been extracted; and to the Berlin Academy of Sciences and Dr Meissel for permission to reprint the tables of J_0 and J_1 which appeared in the *Abhandlungen* for 1888. Dr Meissel has also very generously placed at our disposal the materials for Tables II. and III., the former in manuscript; and Professor J. McMahon has very kindly communicated to us his formulæ for the roots of $J_n(x) = 0$ and other transcendental equations. Our thanks are also especially due to Mr G. A. Gibson, M.A., for his care in reading the proof sheets. Finally we wish to acknowledge our sense of the accuracy with which the text has been set up in type by the workmen of the Cambridge University Press.

The bibliographical list on pp. 289—291 must not be regarded as anything but a list of treatises and memoirs which have been consulted during the composition of this work.

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CORRIGENDUM.

The analysis on p. 39 is invalid if $n > \frac{1}{2}$.

But it holds good if n lies between $-\frac{1}{2}$ and $\frac{1}{2}$, and the semiconvergent expression for J_n satisfies (asymptotically) the relations (16) and (19) on p. 13, so that the formula is applicable for all real values of n.

CHAPTER I.

INTRODUCTORY.

BESSEL'S functions, like so many others, first presented themselves in connexion with physical investigations; it may be well, therefore, before entering upon a discussion of their properties, to give a brief account of the three independent problems which led to their introduction into analysis.

The first of these is the problem of the small oscillations of a uniform heavy flexible chain, fixed at the upper end, and free at the lower, when it is slightly disturbed, in a vertical plane, from its position of stable equilibrium. It is assumed that each element of the string may be regarded as oscillating in a horizontal straight line. Then if m is the mass of the chain per unit of length, l the length of the chain, y the horizontal displacement, at time t, of an element of the chain whose distance from the point of suspension is x, and if T, $T+dT$ are the tensions at the ends of the element, we find, by resolving horizontally,

$$m dx \frac{d^2y}{dt^2} = \frac{d}{dx}\left(T\frac{dy}{dx}\right) dx,$$

or

$$m \frac{d^2y}{dt^2} = \frac{d}{dx}\left(T\frac{dy}{dx}\right).$$

Now, to the degree of approximation we are adopting,

$$T = mg(l-x);$$

and hence

$$\frac{d^2y}{dt^2} = g(l-x)\frac{d^2y}{dx^2} - g\frac{dy}{dx}.$$

If we write z for $(l-x)$, and consider a mode of vibration for which $y = ue^{nti}$, u being a function of z, we shall have

$$z\frac{d^2u}{dz^2} + \frac{du}{dz} + \frac{n^2}{g}u = 0.$$

L 7

Let us put $\kappa^2 = n^2/g$, and assume a solution of the form

$$u = a_0 + a_1 z + a_2 z^2 + \ldots\ldots = \Sigma a_r z^r;$$

then
$$z(2a_2 + 3 . 2 . a_3 z + \ldots\ldots + (r+1) r a_{r+1} z^{r-1} + \ldots\ldots)$$
$$+ (a_1 + 2a_2 z + \ldots\ldots + (r+1) a_{r+1} z^r + \ldots\ldots)$$
$$+ \kappa^2 (a_0 + a_1 z + \ldots\ldots + a_r z^r + \ldots\ldots) = 0,$$

and therefore
$$a_1 + \kappa^2 a_0 = 0,$$
$$4a_2 + \kappa^2 a_1 = 0.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(r+1)^2 a_{r+1} + \kappa^2 a_r = 0;$$

so that
$$u = a_0 \left(1 - \kappa^2 z + \frac{\kappa^4 z^2}{2^2} - \frac{\kappa^6 z^3}{2^2 . 3^2} + \frac{\kappa^8 z^4}{2^2 . 3^2 . 4^2} - \ldots\ldots\right)$$
$$= a_0 \phi(\kappa, z),$$

say.

The series $\phi(\kappa, z)$, as will be seen presently, is a special case of a Bessel function; it is absolutely convergent, and therefore arithmetically intelligible, for all finite values of κ and z.

The fact that the upper end of the chain is fixed is expressed by the condition

$$\phi(\kappa, l) = 0,$$

which, when l is given, is a transcendental equation to find κ, or, which comes to the same thing, n. In other words, the equation $\phi(\kappa, l) = 0$ expresses the influence of the physical data upon the periods of the normal vibrations of the type considered. It will be shown analytically hereafter that the equation $\phi(\kappa, l) = 0$ has always an infinite number of real roots; so that there will be an infinite number of possible normal vibrations. This may be thought intuitively evident, on account of the perfect flexibility of the chain; but arguments of this kind, however specious, are always untrustworthy, and in fact do not prove anything at all.

The oscillations of a uniform chain were considered by Daniel Bernoulli and Euler (*Comm. Act. Petr.* tt. vi, vii, and *Acta Acad. Petr.* t. v.); the next appearance of a Bessel function is in Fourier's *Théorie Analytique de la Chaleur* (Chap. VI.) in connexion with the motion of heat in a solid cylinder.

It is supposed that a circular cylinder of infinite length is heated in such a way that the temperature at any point within it

depends only upon the distance of that point from the axis of the cylinder. The cylinder is then placed in a medium which is kept at zero temperature; and it is required to find the distribution of temperature in the cylinder after the lapse of a time t.

Let v be the temperature, at time t, at a distance x from the axis: then v is a function of x and t. Take a portion of the cylinder of unit length, and consider that part of it which is bounded by cylindrical surfaces, coaxial with the given cylinder, and of radii x, $x + dx$. If K is the conductivity of the cylinder, the excess of the amount of heat which enters the part considered above that which leaves it in the interval $(t, t + dt)$ is

$$\left\{-K\frac{\partial v}{\partial x}\,.\,2\pi x + K\left(2\pi x\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(2\pi x\frac{\partial v}{\partial x}\right)dx\right)\right\}dt;$$

or, say,
$$dH = 2\pi K\left(x\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\right)dxdt.$$

The volume of the part is $2\pi x dx$, so that if D is the density, and C the specific heat, the rise of temperature is $dv = \frac{\partial v}{\partial t}dt$, where
$$CD\,.\,2\pi x dx\frac{\partial v}{\partial t}dt = dH.$$

Hence, by comparison of the two values of dH,

$$CD\frac{\partial v}{\partial t} = K\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x}\right).$$

Fourier writes k for K/CD, and assumes $v = ue^{-nt}$, u being a function of x only; this leads to the differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \frac{n}{k}u = 0; \qquad 2$$

and now, if we put $\frac{n}{k} = g$, we find there is a solution

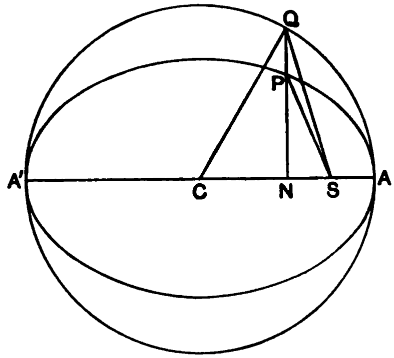
$$u = A\left(1 - \frac{gx^2}{2^2} + \frac{g^2x^4}{2^2\,.\,4^2} - \frac{g^3x^6}{2^2\,.\,4^2\,.\,6^2} + \ldots\ldots\right),$$

which is substantially the same function as that obtained by Bernoulli, except that we have $\frac{1}{4}gx^2$ instead of $\kappa^2 z$.

The boundary condition leads to a transcendental equation to find g; but this is not the place to consider the problem in detail.

Bessel was originally led to the discovery of the functions which bear his name by the investigation of a problem connected with elliptic motion, which may be stated as follows.

Let P be a point on an ellipse, of which AA' is the major axis, S a focus, and C the centre. Draw the ordinate NPQ meeting the auxiliary circle in Q, and join CQ, SP, SQ.



Then in the language of astronomy, the *eccentric anomaly* of P is the number of radians in the angle ACQ, or, which is the same thing, it is ϕ, where

$$\phi = \pi \cdot \frac{\text{area of sector } ACQ}{\text{area of semicircle } AQA'}.$$

It is found convenient to introduce a quantity called the *mean anomaly*, defined by the relation

$$\mu = \pi \cdot \frac{\text{area of elliptic sector } ASP}{\text{area of semi-ellipse } APA'}.$$

(By Kepler's second law of planetary motion, μ is proportional to the time of passage from A to P, supposing that S is the centre of attraction.)

Now by orthogonal projection

$$\begin{aligned} \text{area of } ASP : \text{area of } APA' &= \text{area of } ASQ : \text{area of } AQA' \\ &= (ACQ - CSQ) : AQA' \\ &= (\tfrac{1}{2}a^2\phi - \tfrac{1}{2}ea^2 \sin\phi) : \tfrac{1}{2}\pi a^2 \\ &= (\phi - e\sin\phi) : \pi, \end{aligned}$$

where e is the eccentricity. Hence μ, e, ϕ are connected by the relation

$$\mu = \phi - e \sin \phi. \qquad 3$$

Moreover, if μ and ϕ vary while e remains constant, $\phi - \mu$ is a periodic function of μ which vanishes at A and A'; that is, when μ is a multiple of π. We may therefore assume

$$\phi - \mu = \sum_1^\infty A_r \sin r\mu, \qquad 4$$

and the coefficients A_r are functions of e which have to be determined.

Differentiating 4 with respect to μ, we have

$$\Sigma r A_r \cos r\mu = \frac{d\phi}{d\mu} - 1,$$

and therefore, multiplying by $\cos r\mu$ and integrating,

$$\tfrac{1}{2}\pi r A_r = \int_0^\pi \left(\frac{d\phi}{d\mu} - 1\right) \cos r\mu \, d\mu$$

$$= \int_0^\pi \frac{d\phi}{d\mu} \cos r\mu \, d\mu.$$

Now $\phi = 0$ when $\mu = 0$, and $\phi = \pi$ when $\mu = \pi$; so that by changing the independent variable from μ to ϕ, we obtain

$$\tfrac{1}{2}\pi r A_r = \int_0^\pi \cos r\mu \, d\phi,$$

$$= \int_0^\pi \cos r(\phi - e \sin\phi) \, d\phi,$$

and
$$A_r = \frac{2}{r\pi} \int_0^\pi \cos r(\phi - e \sin\phi) \, d\phi, \qquad 5$$

which is Bessel's expression for A_r as a definite integral. The function A_r can be expressed in a series of positive powers of e, and the expansion may, in fact, be obtained directly from the integral. We shall not, however, follow up the investigation here, but merely show that A_r satisfies a linear differential equation which is analogous to those of Bernoulli and Fourier.

Write x for e, and put

$$u = \frac{\pi r}{2} A_r = \int_0^\pi \cos r(\phi - x \sin\phi) \, d\phi;$$

then, after partial integration of $\frac{du}{dx}$ with respect to ϕ, we find that

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = -r^2 \int_0^\pi \cos r(\phi - x\sin\phi) \, d\phi$$

$$+ \frac{r^2}{x} \int_0^\pi \cos\phi \cos r(\phi - x\sin\phi) \, d\phi$$

$$= -r^2 u - \frac{r^2}{x^2} \int_0^\pi \{(1 - x\cos\phi) - 1\} \cos r(\phi - x\sin\phi) \, d\phi$$

$$= -r^2 u - \frac{r}{x^2} \left[\sin r(\phi - x\sin\phi)\right]_0^\pi + \frac{r^2}{x^2} u$$

$$= -\left(r^2 - \frac{r^2}{x^2}\right) u;$$

or finally
$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + r^2\left(1 - \frac{1}{x^2}\right) u = 0.$$

If we put $rx = z$, this becomes

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{r^2}{z^2}\right)u = 0, \qquad 6$$

and this is what is now considered to be the standard form of Bessel's equation.

If in Fourier's equation,

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \frac{nu}{\kappa} = 0,$$

we put

$$x\sqrt{\frac{n}{\kappa}} = z,$$

the transformed equation is

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + u = 0,$$

which is a special case of Bessel's standard form with $r = 0$.

The differential equation is, for many reasons, the most convenient foundation upon which to base the theory of the functions; we shall therefore define a Bessel function to be a solution of the differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(1 - \frac{n^2}{x^2}\right)u = 0.$$

In the general theory no restriction is placed upon the value of n; the most important case for physical applications is when n is zero or a positive integer. Moreover when n is integral the analytical theory presents some special features; so that for both reasons this case will have to be considered separately.

CHAPTER II.

SOLUTION OF THE DIFFERENTIAL EQUATION.

IF we denote the operation $x\dfrac{d}{dx}$ by ϑ, the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)\,y=0$$

may be written in the form

$$\vartheta^2y+(x^2-n^2)\,y=0.$$

Assume that there is a solution of the form

$$y=x^r(a_0+a_1x+a_2x^2+\ldots)$$
$$=\sum_0^\infty a_sx^{r+s};$$

then, if we substitute this expression in the left-hand side of the differential equation, and observe that $\vartheta x^m=mx^m$, the result is

$$(r^2-n^2)\,a_0x^r+\{(r+1)^2-n^2\}\,a_1x^{r+1}+\sum_2^\infty[\{(r+s)^2-n^2\}\,a_s+a_{s-2}]\,x^{r+s}.$$

The equation will be *formally* satisfied if the coefficient of every power of x in this expression can be made to vanish. Now there is no loss of generality in supposing that a_0 is not zero, hence the first condition to be satisfied is

$$r^2-n^2=0,$$

or

$$r=\pm n.$$

In general, neither of these values of r will make $(r+1)^2-n^2$ vanish*; consequently $a_1=0$, and all the as with odd suffixes must be zero.

* An exception occurs when $n=\frac{1}{2}$, $r=-\frac{1}{2}$; but this does not require separate discussion, since in this case we still have the distinct solutions y_1 and y_2 with $n=\frac{1}{2}$. The only peculiarity is that $r=-\frac{1}{2}$ leads to both of these solutions.

If we take $r = n$, we have

$$s(2n+s)\,a_s + a_{s-2} = 0,$$

and hence
$$a_2 = -\frac{a_0}{2(2n+2)},$$

$$a_4 = -\frac{a_2}{4(2n+4)} = \frac{a_0}{2.4.(2n+2)(2n+4)},$$

and so on. A formal solution of the differential equation is therefore obtained by putting

$$y = y_1 = a_0 x^n \left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} - \dots \right.$$
$$\left. + \frac{(-)^s x^{2s}}{2.4\dots 2s.(2n+2)(2n+4)\dots(2n+2s)} + \dots \right).$$

In a similar way by putting $r = -n$ we obtain the formal solution

$$y_2 = a_0 x^{-n} \left(1 + \frac{x^2}{2(2n-2)} + \frac{x^4}{2.4.(2n-2)(2n-4)} + \dots \right),$$

which, as might be anticipated, only differs from y_1 by the change of n into $-n$.

If n is any finite real or complex quantity, *except a real integer*, the infinite series which occur in y_1 and y_2 are absolutely convergent and intelligible for all finite values of x: each series in fact ultimately behaves like

$$1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \dots$$

the rapid convergence of which is obvious.

The ratio of y_1 to y_2 is not constant; hence (with the same reservation) the general solution of the differential equation is

$$y = Ay_1 + By_2,$$

A and B being arbitrary constants.

If $n = 0$, the integrals y_1 and y_2 are identical; if n is a positive integer y_2 becomes unintelligible, on account of the coefficients in the series becoming infinite. Similarly when n is a negative integer y_2 is still available, but y_1 is unintelligible.

In each of these cases, therefore, it is necessary to discover a second integral; and since n appears only in the form of a square in the differential equation, it will be sufficient to suppose that n is zero or a positive integer.

In accordance with the general theory of linear differential equations, we assume a solution

$$y = (a_0 + b_0 \log x)\, x^{-n} + (a_1 + b_1 \log x)\, x^{-n+1} + \dots$$
$$= x^{-n} \sum_0^\infty (a_s + b_s \log x)\, x^s,$$

then, observing that

$$\vartheta^2 (x^m \log x) = m^2 x^m \log x + 2m x^m,$$

and making a few easy reductions, we find that this form of y gives

$$\begin{aligned}\vartheta^2 y + (x^2 - n^2)\, y = &- 2nb_0 x^{-n} \\ &+ \{(-2n+1)(a_1 + b_1 \log x) + (-2n+2)\, b_1\}\, x^{-n+1} \\ &+ \sum_2^\infty \{s(-2n+s)(a_s + b_s \log x) \\ &+ (a_{s-2} + b_{s-2} \log x) + (-2n+2s)\, b_s\}\, x^{-n+s}.\end{aligned}$$

The expression y will be a formal solution of the differential equation if the coefficient of every term x^{-n+s} or $x^{-n+s} \log x$ on the right-hand side of this identity can be made to vanish. In order that this may be the case it will be found that the following conditions are necessary*:—

(i) The coefficients $b_0, b_1, b_2, \dots b_{2n-1}$ must all vanish.

(ii) All the as with odd suffixes must vanish.

(iii) The coefficient a_0 is indeterminate;

$$a_2 = \frac{a_0}{2(2n-2)},$$

$$a_4 = \frac{a_2}{4(2n-4)} = \frac{a_0}{2 \,.\, 4 \,.\, (2n-2)(2n-4)},$$

and so on, up to

$$a_{2n-2} = \frac{a_0}{2 \,.\, 4 \dots (2n-2)(2n-2)(2n-4) \dots 2}.$$

(iv) The coefficient a_{2n} is indeterminate;

$$b_{2n} = -\frac{a_{2n-2}}{2n},$$

$$b_{2n+2} = -\frac{b_{2n}}{2(2n+2)} = \frac{a_{2n-2}}{2 \,.\, 2n\,(2n+2)},$$

....

$$b_{2n+2s} = (-)^{s-1} \frac{a_{2n-2}}{2 \,.\, 4 \dots 2s \,.\, 2n\,(2n+2) \dots (2n+2s)}, \qquad (s > 0)$$

* The reader will perhaps follow the argument more easily if he will write down a few of the first terms of the sum, and also a few in the neighbourhood of $s = 2n$.

and all the coefficients b_{2n+1}, b_{2n+3}, etc. with odd suffixes must vanish.

(v) Finally

$$a_{2n+2} = -\frac{a_{2n}}{2(2n+2)} - \frac{2n+4}{2(2n+2)} b_{2n+2}$$

$$= -\frac{a_{2n}}{2(2n+2)} + \frac{1}{2(2n+2)}\left\{\frac{1}{2} + \frac{1}{2n+2}\right\} b_{2n};$$

$$a_{2n+4} = -\frac{a_{2n+2}}{4(2n+4)} - \frac{(2n+8)}{4(2n+4)} b_{2n+4}$$

$$= \frac{a_{2n}}{2.4.(2n+2)(2n+4)} - \frac{1}{2.4.(2n+2)(2n+4)} \left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{2n+2} + \frac{1}{2n+4}\right\} b_{2n},$$

and, in general, when $s > 0$,

$$a_{2n+2s} = \frac{(-)^s}{2.4\ldots 2s\,(2n+2)(2n+4)\ldots(2n+2s)} \left\{a_{2n} - b_{2n} \sum_1^s \left(\frac{1}{2s} + \frac{1}{2n+2s}\right)\right\}.$$

All the coefficients which do not vanish may therefore be expressed in terms of two of them; if we choose a_{2n} and b_{2n} for these two, y assumes the form

$$y = a_{2n} y_1 + b_{2n} y_2,$$

where $$y_1 = x^n \left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\ldots\right\} \qquad 7$$

(the solution previously obtained), and

$$y_2 = x^n \left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} - \ldots\ldots\right\} \log x$$

$$- 2^{2n-1} n!(n-1)! \left\{x^{-n} + \frac{x^{-n+2}}{2(2n-2)} + \frac{x^{-n+4}}{2.4.(2n-2)(2n-4)} + \ldots\ldots + \frac{x^{n-2}}{(2.4.6\ldots\overline{2n-2})^2}\right\}$$

$$+ 0x^n + \frac{x^{n+2}}{2(2n+2)}\left\{\frac{1}{2} + \frac{1}{2n+2}\right\}$$

$$- \frac{x^{n+4}}{2.4.(2n+2)(2n+4)}\left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{2n+2} + \frac{1}{2n+4}\right\} + \ldots$$

$$+ (-)^{s-1} \frac{x^{n+2s}}{2.4\ldots 2s\,(2n+2)(2n+4)\ldots(2n+2s)} \sum_1^s \left(\frac{1}{2s} + \frac{1}{2n+2s}\right) + \ldots \qquad 8$$

The characteristic properties of the integral y_2 are that it is the sum of $y_1 \log x$ and a convergent series, proceeding by ascending powers of x, in which only a limited number of negative powers of x occur, and the coefficient of x^n is zero. It becomes infinite, when $x=0$, after the manner of x^{-n}; for any other finite value of x it is finite and calculable, but *not* one-valued, on account of the logarithm which it involves.

The general solution of the differential equation is

$$y = Ay_1 + By_2,$$

A and B being arbitrary constants.

When $n=0$, y_2 does not involve any negative powers of x, and its value may be more simply written

$$y_2 = \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \dots\right) \log x$$
$$+ \frac{x^2}{2^2} - \left(1 + \frac{1}{2}\right) \frac{x^4}{2^2 . 4^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{x^6}{2^2 . 4^2 . 6^2} - \dots \qquad 9$$

It is found convenient, for reasons which will appear as we proceed, to take as the fundamental integrals, when n is a positive integer, not y_1 and y_2 but the quotients of these by $2^n . n!$. These special integrals will be denoted by $J_n(x)$ and $W_n(x)$, so that

$$J_n(x) = \frac{x^n}{2^n . n!}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 . 4 . (2n+2)(2n+4)} - \dots\right\}$$
$$= \sum_0^\infty \frac{(-)^s x^{n+2s}}{2^{n+2s} . s!(n+s)!}, \qquad 10$$

(where, as usual, $0!$ is interpreted to mean 1), and

$$W_n(x) = J_n(x) \log x - \left\{2^{n-1}(n-1)!\, x^{-n} + \frac{2^{n-3}(n-2)!\, x^{-n+2}}{1!} + \frac{2^{n-5}(n-3)!\, x^{-n+4}}{2!} + \dots + \frac{x^{n-2}}{2^{n-1}(n-1)!}\right\}$$
$$+ 0x^n + \sum_1^\infty \left\{\frac{(-)^{s-1} x^{n+2s}}{2^{n+2s} s!(n+s)!} \sum_1^s \left(\frac{1}{2s} + \frac{1}{2n+2s}\right)\right\}. \qquad 11$$

The definition of $J_n(x)$ may be extended to include the case when n is not integral by means of Gauss's function Πn, which is also denoted by $\Gamma(n+1)$. When n is a positive integer, Πn is the same as $n!$; its general value is the limit, when κ is infinite, of

$$\Pi(\kappa, n) = \frac{1 . 2 . 3 \dots \kappa}{(n+1)(n+2)\dots(n+\kappa)} \kappa^n. \qquad 12$$

The function Πn is intelligible and finite for all real finite values of n, except when n is a negative integer, when Πn becomes infinite; by a convention we define $\Pi 0$ to be 1. It will be well to recall the properties of Πn which are expressed by the formulæ

$$\left.\begin{aligned} \Pi n &= n\Pi(n-1) \\ \Pi(-n)\,\Pi(n-1) &= \pi \operatorname{cosec} n\pi \\ \Pi(\tfrac{1}{2}) &= \tfrac{1}{2}\sqrt{\pi} \end{aligned}\right\} \qquad 13$$

(Cf. Forsyth, *Treatise on Differential Equations* (1885) p. 196; Gauss, *Werke* III. p. 144.)

The function Πn may also be interpreted when n is complex; but we shall not require this extension of its meaning.

This being premised, the general definition of $J_n(x)$ is given by the relation

$$J_n(x) = \sum_0^\infty \frac{(-)^s x^{n+2s}}{2^{n+2s}\,\Pi s\,\Pi(n+s)}; \qquad 14$$

and in like manner, if n is not a positive integer,

$$J_{-n}(x) = \sum_0^\infty \frac{(-)^s x^{-n+2s}}{2^{-n+2s}\,\Pi s\,\Pi(-n+s)}. \qquad 14'$$

If n is a positive integer, the function $J_{-n}(x)$, properly speaking, does not exist; it is, however, convenient in this case to adopt the convention* expressed by the formula

$$J_{-n}(x) = (-)^n J_n(x), \ [n \textit{ integral}]. \qquad 15$$

When the argument x remains the same throughout we may write J_n instead of $J_n(x)$, and indicate differentiation with respect to x by accents; thus J_n' will mean $\dfrac{dJ_n(x)}{dx}$, and so on.

By differentiating the general expression for $J_n(x)$ we find that

$$\begin{aligned} J_n' &= \sum_0^\infty \frac{(-)^s (n+2s)\, x^{n+2s-1}}{2^{n+2s}\,\Pi s\,\Pi(n+s)} \\ &= \frac{n}{x} J_n + \sum_1^\infty \frac{(-)^s x^{n+2s-1}}{2^{n+2s-1}\,\Pi(s-1)\,\Pi(n+s)} \\ &= \frac{n}{x} J_n - \sum_0^\infty \frac{(-)^s x^{n+2s+1}}{2^{n+2s+1}\,\Pi s\,\Pi(n+s+1)}, \end{aligned}$$

* It may be proved that if J_{-n} is defined as in (14′) the limiting value of $J_{-(n+\epsilon)}$ when ϵ is infinitesimal, and n a positive integer, is precisely $(-)^n J_n$: so that the convention is necessary in order to secure the continuity of J_n.

(on writing $s+1$ for s); that is,

$$J'_n = \frac{n}{x} J_n - J_{n+1}; \qquad 16$$

or, which is the same thing,

$$J_{n+1} = \frac{n}{x} J_n - J'_n. \qquad 16'$$

Again, writing $n + 2s$ in the form $(n+s)+s$, we have

$$J'_n = \sum_0^\infty \frac{(-)^s x^{n+2s-1}}{2^{n+2s}\,\Pi s\,\Pi(n+s-1)} + \sum_1^\infty \frac{(-)^s x^{n+2s-1}}{2^{n+2s}\,\Pi(s-1)\,\Pi(n+s)}$$

$$= \tfrac{1}{2}(J_{n-1} - J_{n+1}). \qquad 17$$

It will be found that with the convention 15 these formulæ are true for all values of n. It is worth while to notice the special result

$$J'_0 = -J_1. \qquad 18$$

If J_{n+1} is eliminated by combining 16 and 17, the formula

$$J'_n = J_{n-1} - \frac{n}{x} J_n \qquad 19$$

is obtained; and similarly by eliminating J'_n it will be found that

$$J_{n-1} - \frac{2n}{x} J_n + J_{n+1} = 0. \qquad 20$$

The formulæ 16—20 are very important, and are continually required in applications.

It follows from 17 that

$$\begin{aligned} 4J''_n &= 2J'_{n-1} - 2J'_{n+1} \\ &= (J_{n-2} - J_n) - (J_n - J_{n+2}) \\ &= J_{n-2} - 2J_n + J_{n+2}, \end{aligned}$$

and it may be proved by induction that

$$2^s J_n^{(s)} = J_{n-s} - sJ_{n-s+2} + \frac{s(s-1)}{2} J_{n-s+4} - \ldots + (-)^s J_{n+s},$$

the coefficients being those of the binomial theorem for the exponent s.

The analogous formulæ for W_n, which may be obtained by a precisely similar, although more tedious, process, are

$$W_n' = \frac{n}{x} W_n - W_{n+1} + \frac{J_{n+1}}{2(n+1)}, \qquad 21$$

$$W_n' = W_{n-1} - \frac{n}{x} W_n + \frac{J_{n-1}}{2n}, \qquad 22$$

$$W_{n+1} - \frac{2n}{x} W_n + W_{n-1} = \frac{J_{n+1}}{2(n+1)} - \frac{J_{n-1}}{2n}. \qquad 23$$

Now if we put $$2\sigma_n = \sum_1^n \frac{1}{s}, \qquad 24$$

$$Y_n = W_n - \sigma_n J_n, \qquad 25$$

with the convention $\sigma_0 = 0,$

Y_n is a solution of Bessel's equation which is distinct from J_n, and which moreover satisfies the relations

$$Y_n' = \frac{n}{x} Y_n - Y_{n+1}, \qquad 26$$

$$Y_n' = Y_{n-1} - \frac{n}{x} Y_n, \qquad 27$$

$$Y_{n+1} - \frac{2n}{x} Y_n + Y_{n-1} = 0, \qquad 28$$

$$Y_0' = -Y_1, \qquad 29$$

which are of exactly the same form as 16, 19, 20, 18.

The explicit form of Y_n is

$$Y_n = J_n \log x - \left\{ 2^{n-1}(n-1)!\, x^{-n} + \frac{2^{n-3}(n-2)!}{1!} x^{-n+2} + \frac{2^{n-5}(n-3)!}{2!} x^{-n+4} + \dots + \frac{x^{n-2}}{2^{n-1}(n-1)!} \right\}$$
$$- \frac{x^n}{2^{n+1} . n!} \sum_1^n \frac{1}{s} + \sum_1^\infty \frac{(-)^{s-1} k_{n,s} x^{n+2s}}{2^{n+2s} s!\,(n+s)!}, \qquad 30$$

where $$k_{n,s} = \sum_1^s \left(\frac{1}{2s} + \frac{1}{2(n+s)} \right) - \sum_1^n \frac{1}{2s}. \qquad 31$$

The function Y_n was discovered by C. Neumann, and may be referred to as the Neumann function of the n^{th} order. The explicit form 30 was given by Schläfli (*Math. Ann.* III. (1871), p. 143), who obtained it by a different process from that here employed.

There is another way of deducing a second solution of the differential equation which deserves notice. In that equation put $y = uJ_n(x)$, where u is a new dependent variable: then, observing that $J_n(x)$ is an integral, we find that

$$J_n(x)\frac{d^2u}{dx^2} + \left(\frac{1}{x}J_n(x) + 2J_n'(x)\right)\frac{du}{dx} = 0;$$

or

$$\frac{\dfrac{d^2u}{dx^2}}{\dfrac{du}{dx}} + \frac{1}{x} + \frac{2J_n'(x)}{J_n(x)} = 0;$$

whence

$$xJ_n^2(x)\frac{du}{dx} = A,$$

$$u = A\int^x \frac{dx}{xJ_n^2(x)} + B;$$

and

$$y = AJ_n(x)\int^x \frac{dx}{xJ_n^2(x)} + BJ_n(x) \qquad 32$$

is the complete solution of the equation.

By properly choosing the constants A and B it must be possible to make y identical with $J_{-n}(x)$ or $Y_n(x)$ according as n is not or is an integer.

Taking the general value of $J_n(x)$, n not being an integer,

$$xJ_n^2(x) = \frac{1}{2^{2n}(\Pi n)^2}x^{2n+1} + \ldots,$$

so that

$$\int^x \frac{dx}{xJ_n^2(x)} = -2^{2n-1}\Pi n\,\Pi(n-1)\,x^{-2n} + \ldots$$

and

$$AJ_n(x)\int^x \frac{dx}{xJ_n^2(x)} = -2^{n-1}\Pi(n-1)\,Ax^{-n} + \ldots.$$

The leading term of $J_{-n}(x)$ is $\dfrac{2^n}{\Pi(-n)}x^{-n}$, and by making this agree with the preceding, we obtain

$$A = -\frac{2}{\Pi(n-1)\,\Pi(-n)} = -\frac{2}{\pi}\sin n\pi. \qquad 33$$

The value of B will depend upon the lower limit of the integral

$$\int^x \frac{dx}{xJ_n^2(x)}.$$

Divide both sides of 32 by $J_n(x)$ and differentiate with respect to x: then in the case when $y = J_{-n}(x)$ we have

$$\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right) = -\frac{2 \sin n\pi}{\pi x J_n^2},$$

or

$$J_n' J_{-n} - J_{-n}' J_n = \frac{2}{\pi x} \sin n\pi. \qquad 34$$

With the help of 16 and 19 this may be reduced to the form

$$J_n J_{-n+1} + J_{-n} J_{n-1} = \frac{2}{\pi x} \sin n\pi. \qquad 35$$

This suggests a similar formula involving Neumann functions, when n is an integer. If we write

$$u = x(J_{n+1} Y_n - J_n Y_{n+1})$$

we find with the help of 16, 19, 26, 27 that

$$\begin{aligned} \frac{du}{dx} &= J_{n+1} Y_n - J_n Y_{n+1} + Y_n(xJ_n - \overline{n+1} J_{n+1}) \\ &\quad + J_{n+1}(nY_n - xY_{n+1}) - J_n(xY_n - \overline{n+1}\, Y_{n+1}) \\ &\quad - Y_{n+1}(nJ_n - xJ_{n+1}) \\ &= 0 \end{aligned}$$

identically. Hence u is independent of x: and by making use of the explicit forms of J_n and Y_n it is easily found that the value of u is unity. We are thus led to the curious result that

$$J_{n+1} Y_n - J_n Y_{n+1} = \frac{1}{x}. \qquad 36$$

Returning to 32, we find, by comparison with 30, that, when n is a positive integer,

$$Y_n = J_n \int^x \frac{dx}{x J_n^2} + BJ_n. \qquad 37$$

As in the other case, the value of B depends upon the lower limit of the integral.

CHAPTER III.

FUNCTIONS OF INTEGRAL ORDER. EXPANSIONS IN SERIES OF BESSEL FUNCTIONS.

THROUGHOUT this chapter it will be supposed, unless the contrary is expressed, that the parameter n, which occurs in the definition of the Bessel functions, is a positive integer. The expression for the Bessel function of the n^{th} order is

$$J_n(x) = \sum_0^\infty \frac{(-)^s x^{n+2s}}{2^{n+2s} s!\,(n+s)!},$$

which may be written in the form

$$\sum_0^\infty \frac{x^{n+s}}{2^{n+s}.(n+s)!} \cdot \frac{(-x)^s}{2^s.s!};$$

hence we conclude that $J_n(x)$ is the coefficient of t^n in the expansion of $\exp \frac{x}{2}\left(t - \frac{1}{t}\right)$ according to powers of t. In fact,

$$\begin{aligned} \exp \frac{x}{2}(t - t^{-1}) &= \exp \frac{xt}{2} . \exp \frac{-xt^{-1}}{2} \\ &= \sum_0^\infty \frac{x^r t^r}{2^r.r!} . \sum_0^\infty \frac{(-)^s x^s t^{-s}}{2^s.s!} \\ &= \Sigma\Sigma \frac{(-)^s x^{r+s} t^{r-s}}{2^{r+s} r!\ s!}; \end{aligned}$$

and the coefficient of t^n, obtained by putting $r - s = n$, is

$$\Sigma \frac{(-)^s x^{n+2s}}{2^{n+2s}(n+s)!\ s!} = J_n(x).$$

In the same way the coefficient of t^{-n} in the expansion is $(-)^n J_n(x)$, or $J_{-n}(x)$, so that we have identically

$$\exp \frac{x}{2}(t - t^{-1}) = \sum_{-\infty}^{+\infty} J_n(x)\, t^n. \qquad 38$$

The absolute value of $J_{n+1}(x)/J_n(x)$ decreases without limit when n becomes infinite: hence the series on the right hand is absolutely convergent for all finite values of x and t.

Suppose that $t=e^{i\phi}$; then the identity becomes

$$e^{ix\sin\phi} = J_0(x) + 2iJ_1(x)\sin\phi + 2J_2(x)\cos 2\phi + 2iJ_3(x)\sin 3\phi + 2J_4(x)\cos 4\phi + \dots \qquad 39$$

and hence

$$\cos(x\sin\phi) = J_0(x) + 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi + \dots \qquad 40$$

$$\sin(x\sin\phi) = 2J_1(x)\sin\phi + 2J_3(x)\sin 3\phi + 2J_5(x)\sin 5\phi + \dots \qquad 41$$

Change ϕ into $\frac{\pi}{2}-\phi$; thus

$$\cos(x\cos\phi) = J_0(x) - 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi - \dots \qquad 42$$

$$\sin(x\cos\phi) = 2J_1(x)\cos\phi - 2J_3(x)\cos 3\phi + 2J_5(x)\cos 5\phi - \dots \qquad 43$$

These formulæ are true for all finite values of ϕ.

Multiplying the first of these four formulæ by $\cos n\phi$ and integrating from 0 to π, we obtain

$$\int_0^\pi \cos n\phi\cos(x\sin\phi)\,d\phi = \pi J_n(x), \text{ if } n \text{ is even (or zero)},$$

$$= 0, \text{ if } n \text{ is odd},$$

or, in a single formula,

$$\int_0^\pi \cos n\phi\cos(x\sin\phi)\,d\phi = \frac{\pi}{2}\{1+(-1)^n\}\,J_n(x). \qquad 44$$

Similarly

$$\int_0^\pi \sin n\phi\sin(x\sin\phi)\,d\phi = \frac{\pi}{2}\{1-(-1)^n\}\,J_n(x). \qquad 45$$

By addition,

$$\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi = \pi J_n(x), \qquad 46$$

which holds good for all positive integral values of n.

It will be remembered that an integral of this form presented itself in connexion with Bessel's astronomical problem; in fact we arrived at the result that if

$$\mu = \phi - e\sin\phi,$$

then

$$\phi = \mu + \sum_1^\infty A_r \sin r\mu,$$

where

$$A_r = \frac{2}{r\pi}\int_0^\pi \cos r(\phi - e\sin\phi)\,d\phi.$$

We now see that A_r may be written in the form

$$A_r = \frac{2}{r} J_r(re),$$

and in this notation

$$\phi = \mu + 2\left\{J_1(e)\sin\mu + \frac{1}{2}J_2(2e)\sin 2\mu + \frac{1}{3}J_3(3e)\sin 3\mu + \ldots\right\}.$$

It is known that

$$2\cos n\phi = (2\cos\phi)^n - \frac{n}{1}(2\cos\phi)^{n-2} + \frac{n(n-3)}{2!}(2\cos\phi)^{n-4} - \ldots$$

$$+ (-)^s \frac{n(n-s-1)(n-s-2)\ldots(n-2s+1)}{s!}(2\cos\phi)^{n-2s} + \ldots$$

Now let this transformation be applied to the identities 42 and 43, and let the expressions on the left hand be expanded according to powers of $x\cos\phi$: then by equating the coefficients of $\cos^n\phi$ on both sides we find

$$x^n = 2^n . n!\left\{J_n + (n+2)J_{n+2} + \frac{(n+4)(n+1)}{2!}J_{n+4}\right.$$

$$+ \frac{(n+6)(n+2)(n+1)}{3!}J_{n+6} + \ldots$$

$$\left. + \frac{(n+2s)(n+s-1)(n+s-2)\ldots(n+1)}{s!}J_{n+2s} + \ldots\right\}, \quad 47$$

which holds good for all positive integral values of n, and also for $n = 0$. The first three cases are

$$1 = J_0 + 2J_2 + 2J_4 + \ldots + 2J_{2s} + \ldots$$

$$x = 2J_1 + 6J_3 + 10J_5 + \ldots + 2(2s+1)J_{2s+1} + \ldots$$

$$x^2 = 2(4J_2 + 16J_4 + 36J_6 + \ldots + 4s^2J_{2s} + \ldots).$$

In confirmation of these results it should be observed that the series in brackets in 47 is absolutely convergent; for if we write it $\Sigma c_s J_{n+2s}$ we have X J0 =

$$\frac{c_{s+1}J_{n+2s+2}}{c_s J_{n+2s}} = \frac{(n+s)x^2}{4(s+1)(n+2s)(n+2s+1)} \cdot \frac{1 - \dfrac{x^2}{2(2n+4s+6)} + \ldots}{1 - \dfrac{x^2}{2(2n+4s+2)} + \ldots},$$

and this decreases without limit when s increases indefinitely.

Moreover if S_h denotes the sum

$$c_0 J_n + c_1 J_{n+2} + \dots + c_h J_{n+2h},$$

it can be proved by induction that

$$2^n . n! S_h = x^n + \sum_{r=1}^{r=\infty} (-)^{r+h} \frac{(n+h)!(r-1)(r-2)\dots(r-h)}{2^{2r}(n+r+h)!\,r!\,h!} x^{n+2r},$$

or, which is the same thing,

$$2^n . n! S_h = x^n - \frac{(n+h)!\,x^{n+2h+2}}{2^{2h+2}(n+2h+1)!(h+1)!} \left\{1 - \frac{x^2}{2^2(h+2)(n+2h+2).2!} + \dots\right\}.$$

Since the series in brackets is convergent, the expression on the right hand may be made as near to x^n as we please by taking h large enough; moreover the series

$$\sum_{s=h+1}^{s=\infty} c_s J_{n+2s}$$

becomes ultimately infinitesimal when h increases indefinitely; therefore the relation

$$x^n = 2^n . n! \sum_0^\infty c_s J_{n+2s}$$

is true and arithmetically intelligible for all finite values of x.

Now suppose we have an infinite series

$$\sum_0^\infty a_s x^s = a_0 + a_1 x + a_2 x^2 + \dots,$$

then on substituting for each power of x its expression in Bessel functions, and rearranging the terms, we obtain the expression

$$\sum_0^\infty b_s J_s = b_0 J_0 + b_1 J_1 + b_2 J_2 + \dots,$$

where $\qquad b_0 = a_0,\ b_1 = 2a_1,\ b_2 = 8a_2 + 2a_0, \dots,$

and, in general,

$$b_s = 2^s . s! \left\{a_s + \frac{1}{2^2(s-1)} \cdot \frac{a_{s-2}}{1!} + \frac{1}{2^4(s-1)(s-2)} \cdot \frac{a_{s-4}}{2!} + \frac{1}{2^6(s-1)(s-2)(s-3)} \cdot \frac{a_{s-6}}{3!} + \dots\right\}. \quad 48$$

the sum within brackets ending with a term in a_1 or a_0 according as s is odd or even.

If the series $\Sigma a_s x^s$ and $\Sigma b_s J_s$ are both absolutely convergent, we may put

$$\Sigma a_s x^s = \Sigma b_s J_s, \tag{49}$$

and the arithmetical truth of this relation may be verified by a method similar to that employed above.

An important special case is when the series $\Sigma a_s x^s$ satisfies Cauchy's first test of convergence; that is to say, when for all positive integral values of s above a certain limit,

$$\left|\frac{a_s x}{a_{s-1}}\right| < \kappa,$$

where κ is a definitely assigned proper fraction. In this case the limit of $b_s J_s / b_{s-1} J_{s-1}$ is $a_s x / a_{s-1}$, and the series $\Sigma b_s J_s$ is absolutely convergent.

Cases in which $a_s x / a_{s-1}$ is ultimately equal to unity have to be examined separately.

If we are assured of the possibility of an expansion such as that here considered, the coefficients may be found by any method which proves convenient.

As an illustration of this, let us assume

$$Y_0 = J_0 \log x + \Sigma c_n J_n.$$

Then
$$Y_0' = J_0' \log x + \frac{J_0}{x} + \Sigma c_n J_n',$$

$$Y_0'' = J_0'' \log x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \Sigma c_n J_n'';$$

and hence
$$\begin{aligned} 0 &= Y_0'' + \frac{1}{x} Y_0' + Y_0 \\ &= \frac{2J_0'}{x} + \Sigma c_n \left(J_n'' + \frac{1}{x} J_n' + J_n\right) \\ &= \frac{2J_0'}{x} + \Sigma \frac{n^2 c_n}{x^2} J_n. \end{aligned}$$

Therefore
$$\Sigma n^2 c_n J_n = -2xJ_0' = 2xJ_1,$$

by 18. Now by repeated application of 20

$$\begin{aligned} xJ_1 &= 4J_2 - xJ_3 \\ &= 4J_2 - 8J_4 + xJ_5 = \dots \\ &= 2(2J_2 - 4J_4 + 6J_6 - \dots) \end{aligned}$$

Hence $\quad \Sigma n^2 c_n J_n = 4\,(2J_2 - 4J_4 + 6J_6 - \ldots),$
and finally

$$Y_0 = J_0 \log x + 4\left(\frac{1}{2}J_2 - \frac{1}{4}J_4 + \frac{1}{6}J_6 - \ldots\right). \qquad 50$$

Differentiate both sides with regard to x, and apply 17, 18, 29; thus

$$Y_1 = J_1 \log x - \frac{J_0}{x} + 4\left\{\frac{1}{2.2}(J_3 - J_1) - \frac{1}{2.4}(J_5 - J_3) + \frac{1}{2.6}(J_7 - J_5) - \ldots\right\}.$$

that is,

$$Y_1 = J_1 \log x - \frac{J_0}{x} - J_1 + 4\left\{\frac{3}{2.4}J_3 - \frac{5}{4.6}J_5 + \frac{7}{6.8}J_7 - \ldots\right\}. \qquad 51$$

Now we have identically, by 40 or 47,

$$1 = J_0 + 2J_2 + 2J_4 + \ldots,$$

and therefore $\displaystyle -\frac{J_0}{x} = -\frac{1}{x} + \frac{2}{x}J_2 + \frac{2}{x}J_4 + \ldots$

$$= -\frac{1}{x} + \frac{1}{2}(J_1 + J_3) + \frac{1}{4}(J_3 + J_5) + \ldots$$

$$= -\frac{1}{x} + \frac{1}{2}J_1 + \frac{2.3}{2.4}J_3 + \frac{2.5}{4.6}J_5 + \ldots.$$

Consequently

$$Y_1 = J_1 \log x - \frac{1}{x} - \frac{1}{2}J_1 + \frac{6.3}{2.4}J_3 - \frac{2.5}{4.6}J_5 + \frac{6.7}{6.8}J_7 - \ldots + \frac{6\,(4s+3)}{(4s+2)(4s+4)}J_{4s+3} - \frac{2\,(4s+5)}{(4s+4)(4s+6)}J_{4s+5} + \ldots \qquad 52$$

Similarly

$$Y_2 = J_2 \log x - \frac{2J_0}{x^2} - \frac{2J_1}{x} - \frac{3}{2}J_2 + 4\left\{\frac{4J_4}{2.6} - \frac{6J_6}{4.8} + \frac{8J_8}{6.10} - \ldots\right\}, \qquad 53$$

which may be transformed into

$$Y_2 = J_2 \log x - \frac{2}{x^2} - \frac{1}{2}J_0 - \frac{7}{4}J_2 + 5\left(\frac{J_4}{1.3} - \frac{J_6}{2.4}\right) + 9\left(\frac{J_8}{3.5} - \frac{J_{10}}{4.6}\right) + 13\left(\frac{J_{12}}{5.7} - \frac{J_{14}}{6.8}\right) + \ldots, \qquad 54$$

or again into

$$Y_2 = J_2 \log x - \left(\frac{2}{x^2} + \frac{1}{2}\right) - \frac{3}{4} J_2 + \frac{2.4}{1.3} J_4 + \frac{1.3}{2.4} J_6 + \dots$$
$$+ \frac{2s(2s+2)}{(2s-1)(2s+1)} J_{4s} + \frac{(2s-1)(2s+1)}{2s(2s+2)} J_{4s+2} + \dots \qquad 55$$

Neumann has given a general formula for Y_n which may be written in the form

$$Y_n = J_n \log x - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) J_n$$
$$- \frac{n!}{2} \sum_0^{n-1} \frac{1}{n-s} \left(\frac{2}{x}\right)^{n-s} \frac{J_s}{s!} - \sum_1^{\infty} (-)^s \frac{n+2s}{s(n+s)} J_{n+2s}. \qquad 56$$

This may be proved by induction with the help of 20 and 28.

There is another interesting way of expressing Y_n. If we write

$$T_n = 2^{n-1}(n-1)!\, x^{-n} + \frac{2^{n-3}(n-2)!}{1!} x^{-n+2}$$
$$+ \frac{2^{n-5}(n-3)!}{2!} x^{-n+4} + \dots + \frac{x^{n-2}}{2^{n-1}(n-1)!}, \qquad 57$$

a polynomial which has already appeared in the expressions for W_n and Y_n, then we have

$$W_n = J_n \log x - T_n + \frac{n+2}{2(n+1)} \{n+2\} J_{n+2}$$
$$+ \frac{n+4}{4(n+2)} \left\{\frac{n(n+1)}{2!} - 2\right\} J_{n+4}$$
$$+ \frac{n+6}{6(n+3)} \left\{\frac{n(n+1)(n+2)}{3!} + 2\right\} J_{n+6} - \dots$$
$$+ \frac{n+2s}{2s(n+s)} \left\{\frac{n(n+1)\dots(n+s-1)}{s!} + (-)^{s-1} 2\right\} J_{n+2s} + \dots, \qquad 58$$

and the corresponding expression for

$$Y_n = W_n - \frac{1}{2}\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) J_n$$

can at once be written down.

Other illustrations of these expansions will be found in the examples at the end of the book.

A great number of valuable and interesting results are connected with a proposition which may be called the *addition theorem* for Bessel functions; some of these will now be given.

By 38 we have

$$\begin{aligned}\sum_{-\infty}^{+\infty} J_n(u+v)\,.\,t^n &= \exp\left\{\frac{u+v}{2}\left(t-\frac{1}{t}\right)\right\}\\ &= \exp\frac{u}{2}\left(t-\frac{1}{t}\right)\exp\frac{v}{2}\left(t-\frac{1}{t}\right)\\ &= \sum_{-\infty}^{+\infty} J_n(u)\,t^n \sum_{-\infty}^{+\infty} J_n(v)\,t^n.\end{aligned}$$

Multiply out, and equate the coefficients of the powers of t on both sides: thus

$$J_0(u+v) = J_0(u)\,J_0(v) - 2J_1(u)\,J_1(v) + 2J_2(u)\,J_2(v) - \dots$$

$$\begin{aligned}J_1(u+v) = J_0(u)\,J_1(v) &+ J_1(u)\,J_0(v)\\ &- J_1(u)\,J_2(v) - J_2(u)\,J_1(v) + \dots,\end{aligned}$$

and in general

$$\begin{aligned}J_n(u+v) &= \sum_0^n J_s(u)\,J_{n-s}(v)\\ &+ \sum_1^\infty (-)^s\,\{J_s(u)\,J_{n+s}(v) + J_{n+s}(u)\,J_s(v)\}.\end{aligned} \qquad 59$$

Observing that $J_n(-v) = (-)^n J(v)$, we find that if n is odd

$$\begin{aligned}\frac{1}{2}\{J_n(u+v) + J_n(u-v)\} &= J_1(u)\,J_{n-1}(v)\\ &+ J_3(u)\,J_{n-3}(v) + \dots + J_n(u)\,J_0(v)\\ &- J_1(u)\,J_{n+1}(v) - J_3(u)\,J_{n+3}(v) - \dots\\ &+ J_{n+2}(u)\,J_2(v) + J_{n+4}(u)\,J_4(v) + \dots,\end{aligned} \qquad 60$$

and if n is even

$$\begin{aligned}\frac{1}{2}\{J_n(u+v) + J_n(u-v)\} &= J_0(u)\,J_n(v)\\ &+ J_2(u)\,J_{n-2}(v) + \dots + J_n(u)\,J_0(v)\\ &+ J_2(u)\,J_{n+2}(v) + J_4(u)\,J_{n+4}(v) + \dots\\ &+ J_{n+2}(u)\,J_2(v) + J_{n+4}(u)\,J_4(v) + \dots.\end{aligned} \qquad 60'$$

Similarly

if n is odd

$$\begin{aligned}\frac{1}{2}\{J_n(u+v)-J_n(u-v)\} &= J_0(u)\,J_n(v)\\ &\quad + J_2(u)\,J_{n-2}(v)+\ldots+J_{n-1}(u)\,J_1(v)\\ &\quad + J_2(u)\,J_{n+2}(v)+J_4(u)\,J_{n+4}(v)+\ldots\\ &\quad - J_{n+1}(u)\,J_1(v)-J_{n+3}(u)\,J_3(v)-\ldots,\end{aligned} \qquad 61$$

and if n is even

$$\begin{aligned}\frac{1}{2}\{J_n(u+v)-J_n(u-v)\} &= J_1(u)\,J_{n-1}(v)\\ &\quad + J_3(u)\,J_{n-3}(v)+\ldots+J_{n-1}(u)\,J_1(v)\\ &\quad - J_1(u)\,J_{n+1}(v)-J_3(u)\,J_{n+3}(v)-\ldots\\ &\quad - J_{n+1}(u)\,J_1(v)-J_{n+3}(u)\,J_3(v)-\ldots.\end{aligned} \qquad 61'$$

By putting $u=x$ and $v=yi$, where x and y are real quantities, we obtain from these formulæ expressions for the real and imaginary parts of $J_n(x+yi)$. It should be noticed that $J_n(yi)$ immediately presents itself as a real or purely imaginary quantity according as n is even or odd.

We will now consider a remarkable extension of the addition theorem which is due to Neumann*.

By 38 we have

$$\exp\frac{x}{2}\left(kt-\frac{1}{kt}\right)=\sum_{-\infty}^{+\infty}k^nJ_n(x)\,t^n;$$

now
$$\frac{x}{2}\left(kt-\frac{1}{kt}\right)=\frac{kx}{2}\left(t-\frac{1}{t}\right)+\frac{x}{2}\left(k-\frac{1}{k}\right)\frac{1}{t};$$

therefore

$$\exp\frac{x}{2}\left(kt-\frac{1}{kt}\right)=\exp\frac{x}{2t}\left(k-\frac{1}{k}\right).\exp\frac{kx}{2}\left(t-\frac{1}{t}\right),$$

that is,
$$\sum_{-\infty}^{+\infty}k^nJ_n(x)\,t^n=e^{\frac{x}{2t}\left(k-\frac{1}{k}\right)}\sum_{-\infty}^{+\infty}J_n(kx)\,t^n. \qquad 62$$

Put $x=r$, $k=e^{\theta i}$: then

$$\sum_{-\infty}^{+\infty}J_n(re^{\theta i})\,t^n=e^{-\frac{ir\sin\theta}{t}}\sum_{-\infty}^{+\infty}e^{n\theta i}J_n(r)\,t^n. \qquad 63$$

* Strictly speaking, Neumann only considers the case when $n=0$; but the generalisation immediately suggests itself.

Equating the coefficients of t^n, we have

$$J_n(re^{\theta i}) = e^{n\theta i}J_n(r) - \frac{ir\sin\theta}{1!}e^{(n+1)\theta i}J_{n+1}(r) + \frac{i^2r^2\sin^2\theta}{2!}e^{(n+2)\theta i}J_{n+2}(r) - \ldots$$
$$= \xi_n + \eta_n i, \qquad 64$$

where
$$\xi_n = J_n(r)\cos n\theta + r\sin\theta\sin(n+1)\theta J_{n+1}(r) - \frac{r^2\sin^2\theta}{2!}\cos(n+2)\theta J_{n+2}(r) - \frac{r^3\sin^3\theta}{3!}\sin(n+3)\theta J_{n+3}(r) + \ldots, \qquad 64'$$

and
$$\eta_n = J_n(r)\sin n\theta - r\sin\theta\cos(n+1)\theta J_{n+1}(r) - \frac{r^2\sin^2\theta}{2!}\sin(n+2)\theta J_{n+2}(r) + \frac{r^3\sin^3\theta}{3!}\cos(n+3)\theta J_{n+3}(r) + \ldots. \qquad 64''$$

As a special case, let $\theta = \frac{\pi}{2}$; thus

$$J_n(ri) = i^n\left\{J_n(r) + rJ_{n+1}(r) + \frac{r^2}{2!}J_{n+2}(r) + \frac{r^3}{3!}J_{n+3}(r) + \ldots\right\}. \qquad 65$$

Returning to 63 let us put r, θ successively equal to b, β and c, γ and multiply the results together; thus

$$\sum_{-\infty}^{+\infty} J_n(be^{\beta i})t^n \sum_{-\infty}^{+\infty} J_n(ce^{\gamma i})t^n = e^{-\frac{i(b\sin\beta + c\sin\gamma)}{t}} \sum_{-\infty}^{+\infty} e^{n\beta i}J_n(b)t^n \sum_{-\infty}^{+\infty} e^{n\gamma i}J_n(c)t^n. \qquad 66$$

Now the left-hand expression is equal to

$$\sum_{-\infty}^{+\infty} J_n(be^{\beta i} + ce^{\gamma i})t^n,$$

and if the right-hand side is expanded according to powers of t, the coefficient of t^n gives an expression for $J_n(be^{\beta i} + ce^{\gamma i})$ in the form

$$J_n(be^{\beta i} + ce^{\gamma i}) = C_0 - C_1 i(b\sin\beta + c\sin\gamma) + \frac{C_2 i^2(b\sin\beta + c\sin\gamma)^2}{2!} - \ldots, \qquad 67$$

where

$$C_0 = e^{n\beta i} J_n(b)\, J_0(c) + e^{\{(n-1)\beta+\gamma\}i} J_{n-1}(b)\, J_1(c) + \ldots + e^{n\gamma i} J_0(b) J_n(c)$$
$$- e^{\{(n+1)\beta-\gamma\}i} J_{n+1}(b)\, J_1(c) + e^{\{(n+2)\beta-2\gamma\}i} J_{n+2}(b)\, J_2(c) - \ldots$$
$$- e^{\{(n+1)\gamma-\beta\}i} J_1(b)\, J_{n+1}(c) + e^{\{(n+2)\gamma-2\beta\}i} J_2(b) J_{n+2}(c) - \ldots, \quad 68$$

and in like manner for C_1, C_2, etc.

Since, however, this formula is too complicated for practical purposes, we shall only consider in detail the case when $be^{\beta i} + ce^{\gamma i}$ is a real quantity. Moreover we shall suppose in the first instance that $n = 0$.

If we put $$be^{\beta i} + ce^{\gamma i} = a,$$

a real quantity, we have

$$a^2 = (b \cos \beta + c \cos \gamma)^2 + (b \sin \beta + c \sin \gamma)^2$$
$$= b^2 + 2bc \cos(\beta - \gamma) + c^2,$$

and also $$b \sin \beta + c \sin \gamma = 0.$$

Let us put $\beta - \gamma = \alpha$; then the general formula 67 becomes in this special case

$$J_0(\sqrt{b^2 + 2bc \cos \alpha + c^2}) = J_0(b)\, J_0(c) - 2J_1(b)\, J_1(c) \cos \alpha$$
$$+ 2J_2(b)\, J_2(c) \cos 2\alpha - \ldots$$
$$= J_0(b)\, J_0(c) + 2\sum_1^\infty (-)^s J_s(b)\, J_s(c) \cos s\alpha. \quad 69$$

If we change α into $\pi - \alpha$, the formula becomes

$$J_0(\sqrt{b^2 - 2bc \cos \alpha + c^2}) = J_0(b)\, J_0(c) + 2\sum_1^\infty J_s(b)\, J_s(c) \cos s\alpha, \quad 69'$$

and this is Neumann's result already referred to.

By way of verification, put $\alpha = 0$; then we are brought back to the addition formulæ.

Suppose $\alpha = \dfrac{\pi}{2}$; then we have

$$J_0(\sqrt{b^2 + c^2}) = J_0(b)\, J_0(c) - 2J_2(b)\, J_2(c) + 2J_4(b)\, J_4(c) - \ldots \quad 70$$

and hence, by supposing $c = b$,

$$J_0(b\sqrt{2}) = J_0^2(b) - 2J_2^2(b) + 2J_4^2(b) - \ldots. \quad 71$$

In 69 and 69′ suppose $b = c$; thus

$$\left.\begin{aligned} J_0\left(2b\cos\frac{\alpha}{2}\right) &= J_0^2(b) - 2J_1^2(b)\cos\alpha + 2J_2^2(b)\cos 2\alpha - \dots \\ J_0\left(2b\sin\frac{\alpha}{2}\right) &= J_0^2(b) + 2J_1^2(b)\cos\alpha + 2J_2^2(b)\cos 2\alpha + \dots \end{aligned}\right\} \qquad 72$$

In order to obtain another special case of 67 let us suppose that $n = 1$, still retaining the condition $b\sin\beta + c\sin\gamma = 0$. Since the four quantities b, c, β, γ are connected by this single relation, there are three independent quantities at our disposal. Let us choose b, c and $(\beta - \gamma)$, for which, as before, we will write α. Then if a has the meaning already assigned to it, it may be verified, geometrically or otherwise, that

$$ae^{\beta i} = b + ce^{\alpha i}, \quad ae^{\gamma i} = c + be^{-\alpha i}.$$

Now when $n = 1$, the formula 67 gives

$$\begin{aligned} aJ_1(a) = ae^{\beta i}\{J_1(b)\,J_0(c) &+ e^{-\alpha i}J_0(b)\,J_1(c) \\ &- e^{\alpha i}J_2(b)\,J_1(c) + e^{2\alpha i}J_3(b)\,J_2(c) - \dots \\ &- e^{-2\alpha i}J_1(b)\,J_2(c) + e^{-3\alpha i}J_2(b)\,J_3(c) - \dots\}. \end{aligned}$$

Substitute $b + ce^{\alpha i}$ for $ae^{\beta i}$ on the right-hand side, multiply out, and equate the real part of the result to $aJ_1(a)$; thus

$$\begin{aligned} aJ_1(a) &= \{bJ_1(b)\,J_0(c) + cJ_0(b)\,J_1(c)\} \\ &\quad - \{bJ_2(b)\,J_1(c) - bJ_0(b)\,J_1(c) - cJ_1(b)\,J_0(c) + cJ_1(b)\,J_2(c)\}\cos\alpha \\ &\quad + \{bJ_3(b)\,J_2(c) - bJ_1(b)\,J_2(c) - cJ_2(b)\,J_1(c) + cJ_2(b)\,J_3(c)\}\cos 2\alpha \\ &\quad - \dots \\ &= bJ_1(b)\,J_0(c) + cJ_0(b)\,J_1(c) \\ &\quad + \sum_1^\infty (-)^n\{b\,[J_{n+1}(b) - J_{n-1}(b)]\,J_n(c) \\ &\qquad\qquad + c\,[J_{n+1}(c) - J_{n-1}(c)]\,J_n(b)\}\cos n\alpha. \end{aligned} \qquad 73$$

This result may also be obtained from 69 by applying to both sides the operation

$$\delta = b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c},$$

and in fact this operation affords the easiest means of obtaining the formulæ for $n = 2, 3, 4$, etc.

Special results may be obtained from 73 like those deduced from 69. For example, putting $b = c$, we find, after a little reduction, that

$$J_1\left(2b\cos\frac{\alpha}{2}\right) = 2J_0(b)\,J_1(b)\cos\frac{\alpha}{2} - 2J_1(b)\,J_2(b)\cos\frac{3\alpha}{2}$$
$$+ 2J_2(b)\,J_3(b)\cos\frac{5\alpha}{2} - \dots. \qquad 74$$

This may also be deduced from 72 by differentiating with respect to b.

In 72 and 74 write x for b, expand both sides according to powers of $\cos\frac{\alpha}{2}$, and equate coefficients; thus

$$1 = J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) - \dots$$
$$x^2 = 4\,\{J_1^2(x) + 4J_2^2(x) + 9J_3^2(x) + \dots\},$$

and in general

$$x^{2n} = 2^{2n}(n\,!)^2\left\{J_n^2 + \frac{2(n+1)}{1}J_{n+1}^2 + \frac{2(n+2)(2n+1)}{2\,!}J_{n+2}^2 + \dots\right.$$
$$\left. + \frac{2(n+s)(2n+1)(2n+2)\dots(2n+s-1)}{s\,!}J_{n+s}^2 + \dots\right\}, \qquad 75$$

while for the odd powers of x

$$x = \quad 2J_0J_1 + 6J_1J_2 + 10J_2J_3 + \dots + 2(2s+1)\,J_sJ_{s+1} + \dots$$
$$x^3 = 16\left\{J_1J_2 + 5J_2J_3 + 14J_3J_4 + \dots\right.$$
$$\left. + \frac{s(s+1)(2s+1)}{6}J_sJ_{s+1} + \dots\right\},$$

and in general

$$x^{2n-1} = 2^{2n-1}n\,!\ (n-1)!\ \left\{J_{n-1}J_n + (2n+1)\,J_nJ_{n-1}\right.$$
$$+ n(2n+3)\,J_{n+1}J_{n+2} + \frac{2n(2n+5)(2n+1)}{3\,!}J_{n+2}J_{n+3} + \dots$$
$$+ \frac{2n(2n+2s-1)(2n+1)(2n+2)\dots(2n+s-2)}{s\,!}J_{n+s-1}J_{n+s}$$
$$\left. + \dots\dots\right\}. \qquad 76$$

By means of 75 and 76 the series $\sum\limits_0^\infty a_s x^s$ may be transformed into

$$b_0 J_0^2 + b_1 J_0 J_1 + b_2 J_1^2 + b_3 J_1 J_2 + \ldots$$

when $$b_0 = a_0, \quad b_1 = 2a_1, \quad b_2 = 4a_2 + 2a_0,$$

$$b_3 = 16a_3 + 6a_1, \qquad b_4 = 64a_4 + 16a_2 + 2a_0,$$

and so on.

The arguments employed already in a similar case will suffice to prove that we may write

$$\sum_0^\infty a_s x^s = \sum_0^\infty (b_{2s} J_s^2 + b_{2s+1} J_s J_{s+1}), \tag{77}$$

and that this is arithmetically intelligible so long as x remains inside the circle for which $\Sigma a_s x^s$ is absolutely convergent.

This result is also due to Neumann (*Leipzig Berichte* 1869).

We will conclude this chapter by a proof of Schlömilch's theorem, that under certain conditions, which will have to be examined, any function $f(x)$ can be expanded in the form

$$f(x) = \frac{1}{2}a_0 + a_1 J_0(x) + a_2 J_0(2x) + \ldots + a_n J_0(nx) + \ldots, \tag{78}$$

where $$a_0 = \frac{2}{\pi}\int_0^\pi \left\{ f(0) + u\int_0^1 \frac{f'(u\xi)\,d\xi}{\sqrt{1-\xi^2}} \right\} du,$$

and, if $n > 0$,

$$a_n = \frac{2}{\pi}\int_0^\pi u\cos nu \left\{ \int_0^1 \frac{f'(u\xi)\,d\xi}{\sqrt{1-\xi^2}} \right\} du.$$

To prove this, we shall require the lemma

$$\int_0^{\frac{\pi}{2}} J_1(nu\sin\phi)\,d\phi = \frac{1-\cos nu}{nu}. \tag{79}$$

The lemma may be established as follows. We have

$$J_1(nu\sin\phi) = \sum_0^\infty \frac{(-)^s n^{2s+1} u^{2s+1} \sin^{2s+1}\phi}{2^{2s+1} s!(s+1)!};$$

therefore

$$\begin{aligned}\int_0^{\frac{\pi}{2}} J_1(nu\sin\phi)\,d\phi &= \sum_0^\infty \frac{(-)^s n^{2s+1} u^{2s+1}}{2^{2s+1} s!(s+1)!}\int_0^{\frac{\pi}{2}} \sin^{2s+1}\phi\,d\phi \\ &= \sum_0^\infty \frac{(-)^s n^{2s+1} u^{2s+1}}{2^{2s+1} s!(s+1)!}\cdot\frac{2^s s!}{1.3.5\ldots(2s+1)} \\ &= \sum_0^\infty \frac{(-)^s n^{2s+1} u^{2s+1}}{(2s+2)!} = \frac{1-\cos nu}{nu}.\end{aligned}$$

Now if we assume

$$f(x) = \frac{1}{2}a_0 + a_1 J_0(x) + \ldots + a_n J_0(nx) + \ldots$$

we shall have

$$f'(x) = -a_1 J_1(x) - 2a_2 J_1(2x) - \ldots - na_n J_1(nx) - \ldots.$$

Write $u \sin\phi$ for x, and integrate both sides with regard to ϕ between the limits 0 and $\frac{\pi}{2}$; thus

$$\int_0^{\frac{\pi}{2}} f'(u\sin\phi)\,d\phi = -\sum_1^\infty na_n \int_0^{\frac{\pi}{2}} J_1(nu\sin\phi)\,d\phi$$

$$= \Sigma \frac{a_n(\cos nu - 1)}{u};$$

and therefore

$$u\int_0^{\frac{\pi}{2}} f'(u\sin\phi)\,d\phi = \sum_1^\infty a_n \cos nu - \sum_1^\infty a_n$$

$$= \sum_1^\infty a_n \cos nu + \left(\frac{1}{2}a_0 - f(0)\right). \quad 80$$

Hence

$$\frac{1}{2}a_0 - f(0) = \frac{1}{\pi}\int_0^\pi \left\{u\int_0^{\frac{\pi}{2}} f'(u\sin\phi)\,d\phi\right\} du$$

or, which is the same thing,

$$a_0 = \frac{2}{\pi}\int_0^\pi \left\{f(0) + u\int_0^{\frac{\pi}{2}} f'(u\sin\phi)\,d\phi\right\} du,$$

and, when $n > 0$,

$$a_n = \frac{2}{\pi}\int_0^\pi u\cos nu \left\{\int_0^{\frac{\pi}{2}} f'(u\sin\phi)\,d\phi\right\} du.$$

On putting $\sin\phi = \xi$, we obtain the coefficients in the form given by Schlömilch; but it must be observed that the theorem has not yet been proved. All that has been effected is the determination of the coefficients, assuming the possibility of the expansion 78 and also assuming that the result of differentiating the expansion term by term is $f'(x)$.

In order to verify the result *a posteriori*, let the coefficients a_0, a_1, a_2, etc. have the values above assigned to them, and let us write

$$\psi(x) = \frac{1}{2} a_0 + a_1 J_0(x) + a_2 J_0(2x) + \dots$$

$$= f(0) + \frac{2}{\pi}\int_0^\pi u\left(\frac{1}{2} + J_0(x)\cos u + J_0(2x)\cos 2u + \dots\right) F(u)\,du,$$

where
$$F(u) = \int_0^{\frac{\pi}{2}} f'(u\sin\phi)\,d\phi.$$

We have
$$\frac{\pi}{2} J_0(nx) = \int_0^{\frac{\pi}{2}} \cos(nx\sin\phi)\,d\phi$$
$$= \int_0^x \frac{\cos nu\,du}{\sqrt{x^2-u^2}},$$

on putting $x\sin\phi = u$.

Now if x lies between 0 and π, we are entitled to assume an expansion

$$\theta(u) = c_0 + c_1\cos u + c_2\cos 2u + \dots,$$

$\theta(u)$ being equal to $(x^2-u^2)^{-\frac{1}{2}}$ from $u=0$ to $u=x$, and equal to zero from $u=x$ to $u=\pi$; the isolated values $u=0$, x, π being left out of consideration.

By the usual process,

$$\pi c_0 = \int_0^\pi \theta(u)\,du = \int_0^x \frac{du}{\sqrt{x^2-u^2}} = \frac{\pi}{2}.$$

therefore
$$c_0 = \frac{1}{2};$$

while if $n > 0$

$$\frac{\pi}{2} c_n = \int_0^\pi \theta(u)\cos nu\,du = \int_0^x \frac{\cos nu\,du}{\sqrt{x^2-u^2}}$$
$$= \frac{\pi}{2} J_0(nx).$$

Consequently $c_n = J_0(nx)$, and

$$\theta(u) = \frac{1}{2} + J_0(x)\cos u + J_0(2x)\cos 2u + \dots.$$

This is the series which appears in $\psi(x)$, and since the value of $\theta(u)$ is known for the whole range from 0 to π with the exception of an isolated discontinuity when $u = x$, we have

$$\psi(x) = f(0) + \frac{2}{\pi}\int_0^x \frac{uF(u)\,du}{\sqrt{x^2-u^2}}$$

$$= f(0) + \frac{2}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} \frac{f'(u\sin\phi)\,u\,du\,d\phi}{\sqrt{x^2-u^2}}.$$

Put $u\sin\phi = \xi$, $u\cos\phi = \eta$; then this becomes

$$\psi(x) = f(0) + \frac{2}{\pi}\int_0^x \int_0^{\sqrt{x^2-\xi^2}} \frac{f'(\xi)\,d\xi\,d\eta}{\sqrt{x^2-\xi^2-\eta^2}}$$

$$= f(0) + \{f(x) - f(0)\}$$

$$= f(x).$$

This shows that Schlömilch's expansion is valid, provided that x lies between 0 and π, and that the double integral

$$\iint \frac{f'(\xi)\,d\xi\,d\eta}{\sqrt{x^2-\xi^2-\eta^2}}$$

taken over the quadrant of a circle bounded by $\xi = 0$, $\eta = 0$, $\xi^2 + \eta^2 - x^2 = 0$ admits of reduction to $\frac{\pi}{2}\{f(x) - f(0)\}$ which is certainly the case, for instance, if $f(x)$ is finite and continuous over the whole quadrant.

CHAPTER IV.

SEMICONVERGENT EXPANSIONS.

THE power-series which have been obtained for J_n and $Y_n - J_n \log x$ are convergent for all finite values of x, but they become practically useless for numerical calculation when the modulus of x is even moderately large; it is therefore desirable to find expressions which approximate to the true values of J_n and Y_n when x is large, and which admit of easy computation. The expressions which we shall actually obtain are of the form

$$J_n = \sqrt{\frac{2}{\pi x}}\{P \sin x + Q \cos x\}$$

$$Y_n = \sqrt{\frac{\pi}{2x}}\{R \sin x + S \cos x\},$$

where P, Q, R, S are series, in general infinite, proceeding according to descending powers of x. It will appear that these series are ultimately divergent, and the sense in which the equations just written are to be understood is that by taking a suitable number of terms of the expressions on the right we obtain, when x is large, the approximate *numerical* values of J_n and Y_n. The approximate value of J_0 when x is large was discussed by Poisson*, but not in a very detailed or satisfactory manner; in this chapter we shall follow the method of Stokes†, which is important as being capable of application to a large number of functions of a kind which frequently occurs in physical investigations. In a later chapter the question will be discussed in a different manner, depending on the theory of a complex variable.

* *Sur la distribution de la chaleur dans les corps solides.* Journ. de l'École Polyt. cah. 19 (1823), p. 349.

† *On the numerical calculation of a class of definite integrals and infinite series* (Camb. Phil. Trans. IX. (1856; read March 11, 1850), p. 166; or *Collected Papers,* II. p. 329).

On the effect of the internal friction of fluids on the motion of pendulums (Camb. Phil. Trans. IX. (1856; read Dec. 9, 1850), p. [8]).

In Bessel's differential equation put $J_n(x) = ux^{-\frac{1}{2}}$; then it will be found that u is a solution of

$$\frac{d^2u}{dx^2} + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) u = 0. \qquad 81$$

Now when x is large compared with n, the value of $(n^2 - \frac{1}{4})/x^2$ is small; and if, after the analogy of the process employed in the expansion of an implicit function defined by an algebraical equation $f(u, x) = 0$, we omit the term $(n^2 - \frac{1}{4})\,u/x^2$ in the differential equation, we obtain

$$\frac{d^2u}{dx^2} + u = 0,$$

of which the complete solution is

$$u = u_1 = A \sin x + B \cos x,$$

where A, B are constants. We are justified *a posteriori* in regarding this as an approximate solution of 81 because this value of u does in fact make $(n^2 - \frac{1}{4})u/x^2$ small in comparison both with u and with $\dfrac{d^2u}{dx^2}$.

Let us now try to obtain a closer approximation by putting

$$u = u_2 = \left(A_0 + \frac{A_1}{x}\right) \sin x + \left(B_0 + \frac{B_1}{x}\right) \cos x,$$

where A_0, A_1, B_0, B_1 are constants. This value of u gives

$$\frac{d^2u}{dx^2} + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) u = \left\{\frac{2B_1 - (n^2 - \frac{1}{4})A_0}{x^2} - \frac{(n^2 - \frac{9}{4})A_1}{x^3}\right\} \sin x$$
$$- \left\{\frac{2A_1 + (n^2 - \frac{1}{4})B_0}{x^2} + \frac{(n^2 - \frac{9}{4})B_1}{x^3}\right\} \cos x.$$

The expression on the right becomes comparable with x^{-3}, if we assume

$$2A_1 = -(n^2 - \tfrac{1}{4})B_0,$$
$$2B_1 = (n^2 - \tfrac{1}{4})A_0.$$

The value of u_2 thus becomes

$$u_2 = A_0 \left\{\sin x + \frac{n^2 - \frac{1}{4}}{2x} \cos x\right\} + B_0 \left\{\cos x - \frac{n^2 - \frac{1}{4}}{2x} \sin x\right\}$$

and we have

$$\frac{d^2u_2}{dx^2} + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) u_2 = \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2x^3} (-A_0 \cos x + B_0 \sin x).$$

It will be observed that we thus obtain not only an approximate solution when x is large, but an exact solution when $n = \pm \frac{1}{2}$ or $\pm \frac{3}{2}$. We shall return to this second point presently.

Let us now assume

$$u = u_2 + \frac{A_2 \sin x + B_2 \cos x}{x^2};$$

then it will be found that

$$\frac{d^2u}{dx^2} + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) u = -\left\{\frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2} A_0 + 4A_2\right\} \frac{\cos x}{x^3}$$

$$+ \left\{\frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2} B_0 + 4B_2\right\} \frac{\sin x}{x^3}$$

$$- \frac{(n^2 - \frac{25}{4})}{x^4} (A_2 \sin x + B_2 \cos x).$$

If, then, we put

$$A_2 = -\frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2 . 4} A_0,$$

$$B_2 = -\frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2 . 4} B_0,$$

the value of u becomes

$$u = u_3 = A_0 \left\{\sin x + \frac{n^2 - \frac{1}{4}}{2x} \cos x - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2 . 4x^2} \sin x\right\}$$

$$+ B_0 \left\{\cos x - \frac{n^2 - \frac{1}{4}}{2x} \sin x - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2 . 4x^2} \cos x\right\}$$

and we have

$$\frac{d^2u_3}{dx^2} + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) u_3 = \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})(n^2 - \frac{25}{4})}{2 . 4x^4} (A_0 \sin x + B_0 \cos x).$$

Proceeding in this way, we find by induction that, if we put

$$u_r = A_0 U_r + B_0 V_r \qquad 82$$

where $$U_r = \sin x + \frac{n^2 - \frac{1}{4}}{2x} \cos x - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2 . 4x^2} \sin x - \dots$$

$$+ \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4}) \dots (n^2 - (r - \frac{3}{2})^2)}{2 . 4 . 6 \dots (2r - 2) x^{r-1}} \sin \left(x + \frac{r-1}{2} \pi\right), \qquad 83$$

$$V_r = \cos x - \frac{n^2 - \frac{1}{4}}{2x} \sin x - \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4})}{2 . 4x^2} \cos x + \dots$$

$$+ \frac{(n^2 - \frac{1}{4})(n^2 - \frac{9}{4}) \dots (n^2 - (r - \frac{3}{2})^2)}{2 . 4 . 6 \dots (2r - 2) x^{r-1}} \cos \left(x + \frac{r-1}{2} \pi\right), \qquad 84$$

then $\dfrac{d^2u_r}{dx^2}+\left(1-\dfrac{n^2-\frac{1}{4}}{x^2}\right)u_r$

$$=-\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})\dots(n^2-(r-\frac{1}{2})^2)}{2\,.\,4\,.\,6\dots(2r-2)\,x^{r+1}}\left\{A_0\sin\left(x+\frac{r-1}{2}\pi\right)\right.$$
$$\left.+B_0\cos\left(x+\frac{r-1}{2}\pi\right)\right\}. \qquad 85$$

The expression u_r is therefore an approximate solution so long as

$$\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})\dots(n^2-(r-\frac{1}{2})^2)}{2\,.\,4\,.\,6\dots(2r-2)\,x^{r+1}}$$

is small; and u_r is a closer approximation than u_{r-1} so long as $n^2-\left(\dfrac{2r-1}{2}\right)^2$ is numerically less than $2(r-1)x$.

It is to be observed that the closeness of the approximation has been estimated throughout by the numerical smallness of the expression

$$\frac{d^2u_r}{dx^2}+\left(1-\frac{n^2-\frac{1}{4}}{x^2}\right)u_r,$$

the value of which is zero for an exact solution; that it is not to be inferred that the expression above given for u_r reflects in any adequate way the *functional* properties of an exact solution; and finally, that as r is taken larger and larger the expression u_r deviates more and more (in every sense) from a proper solution. These considerations, however, do not debar us from employing the expressions u_r up to a certain point for approximate numerical calculation; and the analysis we have employed shows the exact sense in which u_r is the approximate value of a solution, and the degree of the approximation obtained.

It is convenient to alter the notation by putting

$$A_0\sin x+B_0\cos x=C\cos(\alpha-x),$$
$$A_0\cos x-B_0\sin x=C\sin(\alpha-x), \qquad 86$$

C and α being new constants: this is legitimate, because $C^2=A_0^2+B_0^2$, which is independent of x. Then we have

$$u_r=C\{P_r\cos(\alpha-x)+Q_r\sin(\alpha-x)\}, \qquad 87$$

where

$$P_r = 1 - \frac{(4n^2-1)(4n^2-9)}{1.2(8x)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{1.2.3.4(8x)^4} - \ldots, \qquad 88$$

$$Q_r = \frac{4n^2-1}{8x} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{1.2.3(8x)^3} + \ldots \qquad 89$$

and P_r, Q_r between them contain r terms, involving $x^0, x^{-1}, x^{-2}, \ldots x^{-r+1}$ respectively.

What remains to be done is to determine the constants C and α so that $u_r x^{-\frac{1}{2}}$ may be an approximation to the special solutions of Bessel's equation which are denoted by J_n and J_{-n} or, when n is integral, by J_n and Y_n.

For this purpose we shall require a new expression for J_n in the form of a definite integral. Supposing that $n + \frac{1}{2}$ is real and positive, it may be proved that

$$J_n(x) = \frac{1}{\sqrt{\pi}} \cdot \frac{x^n}{2^n \Pi(n-\frac{1}{2})} \int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d\phi. \qquad 90$$

To show this, we observe that

$$\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d\phi = \int_0^\pi \Sigma \frac{(-)^s x^{2s}\cos^{2s}\phi}{\Pi(2s)} \sin^{2n}\phi\, d\phi$$
$$= \sum_0^\infty (-)^s x^{2s} \frac{\Pi(n-\frac{1}{2})\Pi(s-\frac{1}{2})}{\Pi(2s)\Pi(n+s)}.$$

Now since s is a positive integer

$$\Pi(s-\tfrac{1}{2}) = \frac{(2s-1)(2s-3)\ldots1}{2^s}\Pi(-\tfrac{1}{2}),$$

and
$$\Pi(2s) = 2^s\Pi(s).(2s-1)(2s-3)\ldots1;$$

therefore
$$\frac{\Pi(s-\frac{1}{2})}{\Pi(2s)} = \frac{\Pi(-\frac{1}{2})}{2^{2s}\Pi s} = \frac{\sqrt{\pi}}{2^{2s}\Pi s},$$

and
$$\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d\phi = \sqrt{\pi}\Pi(n-\tfrac{1}{2})\sum_0^\infty \frac{(-)^s x^{2s}}{2^{2s}\Pi s\,\Pi(n+s)}$$
$$= 2^n\sqrt{\pi}\Pi(n-\tfrac{1}{2}).x^{-n}J_n(x),$$

by 14. This proves the proposition.

Now consider the integral

$$U = \int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d\phi = 2\int_0^{\frac{\pi}{2}} \cos(x\cos\phi)\sin^{2n}\phi\, d\phi,$$

and put
$$\cos\phi = 1 - \mu;$$

then
$$d\phi = \frac{d\mu}{\sqrt{2\mu - \mu^2}}$$
and, on writing $\cos x \cos \mu x + \sin x \sin \mu x$ for $\cos(x \cos\phi)$, the integral becomes
$$U = U_1 \cos x + U_2 \sin x,$$
where
$$U_1 = 2\int_0^1 \cos \mu x \,.\, (2\mu - \mu^2)^{n-\frac{1}{2}}\, d\mu,$$
$$U_2 = 2\int_0^1 \sin \mu x \,.\, (2\mu - \mu^2)^{n-\frac{1}{2}}\, d\mu.$$

In the integral U_1 put $\mu x = t$: then it assumes the form
$$U_1 = \frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\int_0^x \cos t \,.\, t^{n-\frac{1}{2}}\left(1 - \frac{t}{2x}\right)^{n-\frac{1}{2}} dt$$
$$= \frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\left\{\int_0^x \cos t \,.\, t^{n-\frac{1}{2}}\, dt + \frac{1}{x}\,.\, V\right\},$$
where
$$V = \int_0^x x\left\{\left(1 - \frac{t}{2x}\right)^{n-\frac{1}{2}} - 1\right\} t^{n-\frac{1}{2}} \cos t\, dt.$$

By breaking up the interval from 0 to x into the intervals $\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \dots \left(\frac{2r-1}{2}\pi, x\right)$, where $(r-1)\pi$ is the greatest multiple of π contained in x, it may be proved that V is finite, however large x may be; hence the most important term in U_1 is
$$\frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\int_0^x t^{n-\frac{1}{2}} \cos t\, dt.$$

This represents the true value more and more nearly the larger x becomes, and at the same time approaches the value
$$\frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\int_0^\infty t^{n-\frac{1}{2}} \cos t\, dt = \frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\,\Pi\left(n - \tfrac{1}{2}\right) \cos\frac{(2n+1)\pi}{4}$$
(supposing that x is positive).

In the same way it may be proved that the approximate value of U_2 when x is very large and positive is
$$\frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\,\Pi\left(n - \tfrac{1}{2}\right) \sin\frac{(2n+1)\pi}{4};$$

and hence, to the same degree of approximation,

$$J_n(x) = \frac{x^n}{2^n \sqrt{\pi}\,\Pi(n-\frac{1}{2})} \{U_1 \cos x + U_2 \sin x\}$$

$$= \sqrt{\frac{2}{\pi x}} \cos\left\{\frac{(2n+1)\pi}{4} - x\right\}. \qquad 91$$

Comparing this with 87 we see that

$$C = \sqrt{\frac{2}{\pi}}, \qquad \alpha = \frac{(2n+1)\pi}{4},$$

and hence that

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \left[P \cos\left\{\frac{(2n+1)\pi}{4} - x\right\} \right.$$

$$\left. + Q \sin\left\{\frac{(2n+1)\pi}{4} - x\right\}\right], \qquad 92$$

with $P = 1 - \frac{(4n^2-1)(4n^2-9)}{2!\,(8x)^2}$

$$+ \frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{4!\,(8x)^4} - \dots,$$

$$Q = \frac{4n^2-1}{8x} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!\,(8x)^3} + \dots.$$

For the sake of reference it is convenient to give in this place the corresponding expression for Y_n which may be deduced from a result of Hankel's (*Math. Ann.* I. (1869), pp. 471, 494) to be considered later on.

If we write γ for Euler's constant, which is otherwise denoted by $-\psi(0)$ or $-\Pi'(0) \div \Pi(0)$, and the value of which is

$$\gamma = -\psi(0) = \cdot 57721\ 56649\ 01532\ 86060\ 65\dots \qquad 93$$

and if log 2 is the natural logarithm of 2, the value of which is

$$\log 2 = \cdot 69314\ 71805\ 59945\ 30941\ 72\dots, \qquad 94$$

then the approximate value of Y_n is, when x is a large real positive quantity,

$$Y_n(x) = (\log 2 - \gamma) J_n(x) + \sqrt{\frac{\pi}{2x}} \left[Q \cos\left\{\frac{(2n+1)\pi}{4} - x\right\}\right.$$

$$\left. - P \sin\left\{\frac{(2n+1)\pi}{4} - x\right\}\right], \qquad 95$$

the values of $J_n(x)$, P, and Q on the right hand being those given above.

Thus to the first degree of approximation

$$Y_n = \sqrt{\frac{2}{\pi x}}(\log 2 - \gamma)\cos\left\{\frac{(2n+1)\pi}{4} - x\right\} - \sqrt{\frac{\pi}{2x}}\sin\left\{\frac{(2n+1)\pi}{4} - x\right\}$$

$$= (\log 2 - \gamma) J_n + \frac{\pi}{2} J_{n+1}. \qquad 96$$

The value of $\log 2 - \gamma$ is, to twenty-two places,

$$\log 2 - \gamma = \cdot 11593\ 15156\ 58412\ 44881\ 07. \qquad 97$$

It is interesting to confirm these results by means of the relation $J_{n+1} Y_n - J_n Y_{n+1} = \frac{1}{x}$.

The approximate formulæ for J_n and Y_n hold good for all values of x with a large modulus, provided the real part of x is positive, and that value of $\sqrt{x}$ is taken which passes continuously into a real positive value when x is real and positive. The case when x is a pure imaginary will have to be considered separately in a subsequent chapter.

We will now briefly consider the case, already alluded to, when n is the half of an odd integer.

By the general definition of $J_n(x)$, we have

$$J_{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\,\Pi(-\frac{1}{2})}\left\{1 - \frac{x^2}{2 \,.\, 1} + \frac{x^4}{2 \,.\, 4 \,.\, 1 \,.\, 3} - \dots\right\}$$

$$= \sqrt{\frac{2}{\pi x}}\cos x,$$

and

$$J_{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\,\Pi(\frac{1}{2})}\left\{1 - \frac{x^2}{2 \,.\, 3} + \frac{x^4}{2 \,.\, 4 \,.\, 3 \,.\, 5} - \dots\right\}$$

$$= \sqrt{\frac{2x}{\pi}} \cdot \frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}}\sin x.$$

Hence and by means of the relation

$$J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$$

we may calculate the expression for $J_{k+\frac{1}{2}}$ where k is any positive or negative integer. The functions thus obtained are of importance in certain physical applications, so that the following short table may be useful.

$2n$	$J_n \times \sqrt{\frac{1}{2}\pi x}$
1	$\sin x$
3	$\frac{\sin x}{x} - \cos x$
5	$\left(\frac{3}{x^2} - 1\right) \sin x - \frac{3}{x} \cos x$
7	$\left(\frac{15}{x^3} - \frac{6}{x}\right) \sin x - \left(\frac{15}{x^2} - 1\right) \cos x$
9	$\left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right) \sin x - \left(\frac{105}{x^3} - \frac{10}{x}\right) \cos x$
11	$\left(\frac{945}{x^5} - \frac{420}{x^3} + \frac{15}{x}\right) \sin x - \left(\frac{945}{x^4} - \frac{105}{x^2} + 1\right) \cos x$
-1	$\cos x$
-3	$-\sin x - \frac{\cos x}{x}$
-5	$\frac{3}{x} \sin x + \left(\frac{3}{x^2} - 1\right) \cos x$
-7	$-\left(\frac{15}{x^2} - 1\right) \sin x - \left(\frac{15}{x^3} - \frac{6}{x}\right) \cos x$
-9	$\left(\frac{105}{x^3} - \frac{10}{x}\right) \sin x + \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right) \cos x$
-11	$-\left(\frac{945}{x^4} - \frac{105}{x^2} + 1\right) \sin x - \left(\frac{945}{x^5} - \frac{420}{x^3} + \frac{15}{x}\right) \cos x$

These expressions may be derived from 92 by assigning the proper value to n; the series P and Q terminate when $2n$ is an odd integer, and we thus see how it is that the analysis employed in finding the approximate value of J_n leads to the exact value in this special case.

The reader who is acquainted with the modern theory of linear

differential equations will observe that the "indicial equation" corresponding to

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{n^2}{x^2}\right)u=0$$

is

$$r^2-n^2=0;$$

so that when $2n$ is an odd integer we have the case when the roots of the indicial equation, although not themselves integral, differ by an integer; and it is this circumstance that gives rise to the exceptional character of the solutions. (Cf. Craig, *Theory of Linear Differential Equations*, vol. I. (1889), chap. 5.)

CHAPTER V.

THE ZEROES OF THE BESSEL FUNCTIONS.

For the purpose of realising the general behaviour of a transcendental function it is important to discover, if possible, the values of the independent variable which cause the function to vanish or become infinite or to assume a maximum or minimum value. It has already been observed that the function $J_n(x)$ is finite and continuous for all finite values of x; we will now proceed to investigate the zeroes of the function, that is to say, the values of x which make $J_n(x) = 0$. It will be supposed in the first place that n is a positive integer, or zero, this being by far the most important case for physical applications.

On account not only of its historical interest, but of its directness and simplicity, we reproduce here Bessel's original proof of the theorem that the equation $J_n(x) = 0$ has an infinite number of real roots*.

It has been shown (see 46 above) that

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin\phi)\, d\phi$$
$$= \frac{2}{\pi}\int_0^1 \cos(xt) \frac{dt}{\sqrt{1-t^2}}.$$

Suppose that $x = \dfrac{2m+m'}{2}\pi$ where m is a positive integer, and m' a positive proper fraction; then

$$J_0(x) = \frac{2}{\pi}\int_0^1 \cos\frac{(2m+m')\pi t}{2} \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{2}{\pi}\int_0^{2m+m'} \cos\frac{\pi v}{2} \frac{dv}{\sqrt{\{(2m+m')^2 - v^2\}}}.$$

* Berlin Abhandlungen (1824), Art. 14.

Now $\int_a^b \cos\frac{\pi v}{2}\frac{dv}{\sqrt{\{(2m+m')^2-v^2\}}}$

$$=\int_{a-h}^{b-h}\cos\left(\frac{h\pi}{2}+\frac{\pi u}{2}\right)\frac{du}{\sqrt{\{(2m+m')^2-(h+u)^2\}}},$$

and hence, by taking $a=h-1$, $b=h+1$, and putting h successively equal to 1, 3, 5, ... $(2m-1)$, we obtain, on writing μ for $2m+m'$,

$$J_0(x)=\frac{2}{\pi}\int_{-1}^{+1}\sin\frac{\pi u}{2}\left\{\frac{-1}{\sqrt{\mu^2-(1+u)^2}}+\frac{1}{\sqrt{\mu^2-(3+u)^2}}-\dots\right.$$
$$\left.+\frac{(-1)^m}{\sqrt{\mu^2-(2m-1+u)^2}}\right\}du$$
$$+\frac{2}{\pi}(-1)^m\int_0^{m'}\cos\frac{\pi u}{2}\frac{du}{\sqrt{\mu^2-(2m+u)^2}}.$$

The integral

$$\int_{-1}^{+1}\sin\frac{\pi u}{2}\frac{du}{\sqrt{\mu^2-(h+u)^2}}$$
$$=\int_0^1\sin\frac{\pi u}{2}\left\{\frac{1}{\sqrt{\mu^2-(h+u)^2}}-\frac{1}{\sqrt{\mu^2-(h-u)^2}}\right\}du,$$

and is therefore positive; moreover

$$\int_0^{m'}\cos\frac{\pi u}{2}\frac{du}{\sqrt{\mu^2-(2m+u)^2}}$$

is evidently positive. Therefore $J_0(x)$ has been reduced to the form

$$J_0(x)=-u_1+u_2-u_3+\dots+(-)^m u_m,$$

where $u_1, u_2, \dots u_m$ are a series of positive quantities and

$$u_1<u_2<u_3\dots<u_m.$$

Therefore $J_0(x)$ is positive or negative according as m is even or odd; and consequently as x increases from $k\pi$ to $(k+1)\pi$ where k is any positive integer (or zero) $J_0(x)$ changes sign, and must therefore vanish for some value of x in the interval.

This proves that the equation $J_0(x)=0$ has an infinite number of real positive roots; the negative roots are equal and opposite to the positive roots.

It has been shown in the last chapter that the asymptotic value of $J_0(x)$ is

$$\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}\right),$$

so that the large roots of $J_0(x)$ are approximately given by

$$x = (k + \tfrac{3}{4})\pi,$$

where k is a large positive or negative integer.

To give an example of the degree of the approximation, suppose $k = 9$; then

$$(k + \tfrac{3}{4})\pi = 30{\cdot}6305\ldots,$$

the true value of the corresponding root being

$$30{\cdot}6346\ldots.$$

It has been proved that

$$J_1(x) = -\frac{dJ_0(x)}{dx};$$

now between every two successive roots of $J_0(x)$ the derived function $J_0'(x)$ must vanish at least once, and therefore between every pair of adjacent roots of $J_0(x)$ there must be at least one root of $J_1(x)$. Thus the equation $J_1(x) = 0$ has also an infinite number of real roots; we infer from the asymptotic value of $J_1(x)$ that the large roots are approximately given by

$$x = (k + \tfrac{1}{4})\pi,$$

where k, as before, is any large integer.

Now let n be any positive integer, and put

$$\left(\frac{x}{2}\right)^{-n} J_n(x) = R_n,$$

$$\frac{x^2}{4} = \xi;$$

then it may be verified from 16 or 10 that

$$R_{n+1} = -\frac{dR_n}{d\xi},$$

so that R_{n+1} vanishes when R_n is a maximum or a minimum. We infer, as before, that between any two consecutive roots of $R_n = 0$ there must be at least one root of $R_{n+1} = 0$; and hence, by induction, that the equation $J_n(x) = 0$ has an infinite number of real roots.

The existence of the real roots of $J_n(x)$ having thus been proved, it remains for us to devise a method of calculating them. We will begin by explaining a process which, although of little or no practical value, is very interesting theoretically. Let ξ and R_n have the meanings just given to them, and put

$$R_n = f_s(\xi) + \phi_s(\xi),$$

where $f_s(\xi)$ is the sum of the first s terms of R_n, and $\phi_s(\xi)$ is the rest of the series. Then when s is very large $\phi_s(\xi)$ is very small for all finite values of ξ, and therefore, if β is a root of R_n, $f_s(\beta)$ must also be very small for a sufficiently large value of s. For large values of s, therefore, there will be real roots of the algebraical equation $f_s(\xi)=0$ which are approximations to the roots of $R_n=0$; and even for moderate values of s, we may expect to obtain approximations from $f_s(\xi)=0$, but we must be careful to see that $\phi_s(\beta)$ is small as well as $f_s(\beta)=0$.

If the real roots of the equations $f_s=0$ are plotted off, it will be found that they ultimately fall into groups or clusters, each cluster "condensing" in the neighbourhood of a root of $R_n=0$. The points belonging to a particular cluster are, of course, derived from different equations $f_s=0$.

As an illustration, suppose $n=0$; then

$$R_0 = 1 - \xi + \frac{\xi^2}{1^2 . 2^2} - \frac{\xi^3}{1^2 . 2^2 . 3^2} + \frac{\xi^4}{1^2 . 2^2 . 3^2 . 4^2} - \dots,$$

and the equations $f_s=0$ are

$$\begin{aligned} \xi - 1 &= 0 \\ \xi^2 - 4\xi + 4 &= 0 \\ \xi^3 - 9\xi^2 + 36\xi - 36 &= 0 \\ \xi^4 - 16\xi^3 + 144\xi^2 - 576\xi + 576 &= 0, \end{aligned}$$

and so on.

The real roots of these are, in order,

1

2, 2

1·42999...

1·44678..., 5·42...

and we already see the beginning of a condensation in the succession 1·42999, 1·44678. If we put

$$\frac{x^2}{4} = 1{\cdot}44678,$$

we find $$x = 2{\cdot}405\dots;$$

and the least positive root of $J_0(x)=0$ is, in fact,

$$x_1 = 2{\cdot}4048\dots$$

so that $x = x_1$ to three places of decimals.

The best practical method of calculating the roots is that of Stokes*, which depends upon the semiconvergent expression for $J_n(x)$.

To fix the ideas, suppose $n = 0$. Then we have approximately

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left\{P\cos\left(x - \frac{\pi}{4}\right) + Q\sin\left(x - \frac{\pi}{4}\right)\right\},$$

where $$P = 1 - \frac{1^2 . 3^2}{2!(8x)^2} + \frac{1^2 . 3^2 . 5^2 . 7^2}{4!(8x)^4} - \ldots,$$

$$Q = \frac{1}{8x} - \frac{1^2 . 3^2 . 5^2}{3!(8x)^3} + \frac{1^2 . 3^2 . 5^2 . 7^2 . 9^2}{5!(8x)^5} - \ldots.$$

Put $$P = M\cos\psi,\ Q = M\sin\psi;$$

then $$M = \sqrt{P^2 + Q^2},\quad \psi = \tan^{-1}\frac{Q}{P}.$$

Now it is not difficult to see that when x is so large that we may use a few terms of P and Q to find an approximate value for $J_0(x)$, we are justified in calculating $\sqrt{P^2 + Q^2}$ and $\tan^{-1}\frac{Q}{P}$ as if P and Q were convergent series; the results, of course, will have the same kind of meaning as the series P and Q from which they are derived. We thus obtain the semiconvergent expressions

$$\left.\begin{aligned} M &= 1 - \frac{1}{16x^2} + \frac{53}{512x^4} - \ldots, \\ \psi &= \tan^{-1}\left(\frac{1}{8x} - \frac{33}{512x^3} + \frac{3417}{16384x^5} - \ldots\right). \end{aligned}\right\} \quad 98$$

The value of $J_0(x)$ is (approximately)

$$\sqrt{\frac{2}{\pi x}}\,M\cos\left(x - \frac{\pi}{4} - \psi\right), \quad 99$$

which vanishes when

$$x = (k - \tfrac{1}{4})\pi + \psi,$$

k being any integer.

Write, for the moment,

$$\phi = (k - \tfrac{1}{4})\pi;$$

* Camb. Phil. Trans. IX. (1856), p. 182.

then we have to solve the transcendental equation

$$x = \phi + \tan^{-1}\left(\frac{1}{8x} - \frac{33}{512x^3} + \frac{3417}{16384x^5} - \dots\right)$$

on the supposition that ϕ and x are both large.

Assume $$x = \phi + \frac{a}{\phi} + \frac{b}{\phi^3} + \frac{c}{\phi^5} + \dots;$$

then, with the help of Gregory's series,

$$\phi + \frac{a}{\phi} + \frac{b}{\phi^3} + \frac{c}{\phi^5} + \dots$$

$$= \phi + \frac{1}{8}\left(\frac{1}{\phi} - \frac{a}{\phi^3} + \dots\right) - \frac{33}{512}\left(\frac{1}{\phi^3} - \dots\right) + \dots$$

$$- \frac{1}{3.512}\left(\frac{1}{\phi^3} - \dots\right) + \dots$$

and therefore $$a = \frac{1}{8}, \quad b = -\frac{31}{384}, \text{ etc.}$$

Substituting for ϕ its value $(k - \frac{1}{4})\pi$, we have finally

$$\frac{x}{\pi} = (k - \tfrac{1}{4}) + \frac{1}{2\pi^2(4k-1)} - \frac{31}{6\pi^4(4k-1)^3} + \dots,$$

or, reducing to decimals,

$$\frac{x}{\pi} = k - {\cdot}25 + \frac{{\cdot}050661}{4k-1} - \frac{{\cdot}053041}{(4k-1)^3} + \frac{{\cdot}262051}{(4k-1)^5} - \dots. \qquad 100$$

The corresponding formula for the roots of $J_1(x) = 0$ is

$$\frac{x}{\pi} = k + {\cdot}25 - \frac{{\cdot}151982}{4k+1} + \frac{{\cdot}015399}{(4k+1)^3} - \frac{{\cdot}245270}{(4k+1)^5} + \dots. \qquad 100'$$

The same method is applicable to Bessel functions of higher orders.

The general formula for the kth root of $J_n(x) = 0$ is

$$\left.\begin{aligned} x = a - \frac{m-1}{8a} - \frac{4(m-1)(7m-31)}{3(8a)^3} \\ - \frac{32(m-1)(83m^2 - 982m + 3779)}{15(8a)^5} + \dots, \\ \text{where} \quad a = \tfrac{1}{4}\pi(2n - 1 + 4k), \qquad m = 4n^2. \end{aligned}\right\} \qquad 100''$$

This formula is due to Prof. McMahon, and was kindly communicated to the authors by Lord Rayleigh. It has been worked out independently by Mr W. St B. Griffith, so that there is no reasonable doubt of its correctness. It may be remarked that Stokes gives the incorrect value ·245835 for the numerator of the last term on the right-hand side of 10; the error has somehow arisen in the reduction of $1179/5\pi^6$, which is the exact value, to a decimal.

The values of the roots may also be obtained by interpolation from a table of the functions, provided the tabular difference is sufficiently small.

The reader will find at the end of the book a graph of the functions $J_0(x)$ and $J_1(x)$ extending over a sufficient interval to show how they behave when x is comparatively large.

It seems probable that between every pair of successive real roots of $J_n(x)$ there is exactly one real root of $J_{n+1}(x)$. It does not appear that this has been strictly proved; there must in any case be an odd number of roots in the interval.

With regard to values of n which are not integral, it will be sufficient for the present to state that if n is real the equation $J_n(x) = 0$ has an infinite number of real roots; and if $n > -1$, all the roots of the equation are real.

CHAPTER VI.

FOURIER-BESSEL EXPANSIONS.

ONE of the most natural ways in which the Bessel functions present themselves is in connexion with the theory of the potential. This has, in fact, already appeared in the introduction (p. 3); we will now consider this part of the theory in some detail, adopting, in the main, the method of Lord Rayleigh (*Phil. Mag.* November 1872).

If we use cylindrical coordinates r, θ, z, Laplace's equation $\nabla^2\phi = 0$, which must be satisfied by a potential function ϕ, becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \qquad 101$$

Assume
$$\phi = ue^{-\kappa z}\cos n\theta, \qquad 102$$

where κ is a real positive quantity, and u is independent of θ and z. Then ϕ will satisfy Laplace's equation if

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\kappa^2 - \frac{n^2}{r^2}\right)u = 0; \qquad 103$$

whence
$$\left.\begin{aligned} u &= AJ_n(\kappa r) + BJ_{-n}(\kappa r) \\ \text{or}\quad u &= AJ_n(\kappa r) + BY_n(\kappa r) \end{aligned}\right\} \qquad 104$$

according as n is not or is an integer.

Consider the two particular solutions

$$\left.\begin{aligned} \phi &= e^{-\kappa z}\cos n\theta J_n(\kappa r) \\ \psi &= e^{-\lambda z}\cos n\theta J_n(\lambda r); \end{aligned}\right\} \qquad 105$$

then since $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$, Green's theorem gives

$$\iint \phi \frac{\partial \psi}{\partial \nu} dS = \iint \psi \frac{\partial \phi}{\partial \nu} dS \qquad 106$$

the surface integration being taken over the boundary of a closed space throughout which ϕ and ψ are finite, continuous, and one-valued; as usual, $\frac{\partial \phi}{\partial \nu}$ and $\frac{\partial \psi}{\partial \nu}$ denote the space rates of variation of ϕ and ψ along the outward normal to the element dS.

First, suppose that n is an integer, and that κ and λ are positive. Then we may integrate over the surface bounded by the cylinder $r = a$, and the planes $z = 0$, $z = +\infty$.

When $z = 0$,

$$\phi = \cos n\theta J_n(\kappa r), \quad \frac{\partial \psi}{\partial \nu} = \lambda \cos n\theta J_n(\lambda r),$$

and the part of the integral on the left-hand side of 106 which is derived from the flat circular end bounded by $z = 0$, $r = a$, is

$$\lambda \int_0^{2\pi} \int_0^a r\, d\theta\, dr \cos^2 n\theta J_n(\kappa r) J_n(\lambda r) = \pi\lambda \int_0^a J_n(\kappa r) J_n(\lambda r) r\, dr.$$

When $z = +\infty$, $\phi \frac{\partial \psi}{\partial \nu}$ vanishes, and nothing is contributed to the integral.

For the curved surface $r = a$,

$$\phi = e^{-\kappa z} \cos n\theta J_n(\kappa a), \quad \frac{\partial \psi}{\partial \nu} = \lambda e^{-\lambda z} \cos n\theta J_n'(\lambda a),$$

so that the corresponding part of the integral is

$$\lambda J_n(\kappa a) J_n'(\lambda a) \int_0^{2\pi} \int_0^\infty a\, d\theta\, dz e^{-(\kappa+\lambda)z} \cos^2 n\theta$$

$$= \frac{\pi\lambda a}{\kappa + \lambda} J_n(\kappa a) J_n'(\lambda a).$$

Working out the other side of 106 in a similar way, we obtain

$$\lambda \int_0^a J_n(\kappa r) J_n(\lambda r) r\, dr + \frac{\lambda a}{\kappa + \lambda} J_n(\kappa a) J_n'(\lambda a)$$

$$= \kappa \int_0^a J_n(\kappa r) J_n(\lambda r) r\, dr + \frac{\kappa a}{\kappa + \lambda} J_n'(\kappa a) J_n(\lambda a)$$

or, finally

$$(\kappa - \lambda)\int_0^a J_n(\kappa r)\, J_n(\lambda r)\, r\, dr$$

$$= \frac{a}{\kappa + \lambda}\{\lambda J_n(\kappa a)\, J_n'(\lambda a) - \kappa J_n'(\kappa a)\, J_n(\lambda a)\}. \qquad 107$$

The very important conclusion follows that if κ and λ are different,

$$\int_0^a J_n(\kappa r)\, J_n(\lambda r)\, r\, dr = 0, \qquad 108$$

provided that

$$\lambda J_n(\kappa a)\, J_n'(\lambda a) - \kappa J_n'(\kappa a)\, J_n(\lambda a) = 0. \qquad 108'$$

The condition 108′ is satisfied, among other ways,

(i) if κ, λ are different roots of $J_n(ax) = 0$,

(ii) if they are different roots of $J_n'(ax) = 0$,

(iii) if they are different roots of

$$AxJ_n'(ax) + BJ_n(ax) = 0,$$

where A and B are independent of x. All these cases occur in physical applications.

In the formula 107 put $\kappa = \lambda + h$, divide both sides by h, and make h decrease indefinitely; then we find, with the help of Taylor's theorem, that

$$\int_0^a J_n^2(\lambda r)\, r\, dr$$

$$= \frac{a}{2\lambda}\{\lambda a\,[J_n'(\lambda a)]^2 - J_n(\lambda a)\, J_n'(\lambda a) - \lambda a J_n(\lambda a)\, J_n''(\lambda a)\}.$$

Reducing this with the help of the differential equation, we obtain finally

$$\int_0^a J_n^2(\lambda r)\, r\, dr = \frac{a^2}{2}\left\{J_n'^2(\lambda a) + \left(1 - \frac{n^2}{\lambda^2 a^2}\right) J_n^2(\lambda a)\right\}. \qquad 109$$

When n is not an integer, we may still apply Green's theorem, provided that in addition to the cylindrical surface already considered we construct a diaphragm extending from the axis to the circumference in the plane $\theta = 0$, and consider this as a *double* boundary, first for $\theta = 0$, and then for $\theta = 2\pi$.

When $\theta = 0$,

$$\phi = e^{-\kappa z} J_n(\kappa r), \quad \frac{\partial \psi}{\partial \nu} = 0;$$

and when $\theta = 2\pi$,

$$\phi = e^{-\kappa z} \cos 2n\pi J_n(\kappa r), \quad \frac{\partial \psi}{\partial \nu} = -\frac{n}{r} e^{-\lambda z} \sin 2n\pi J_n(\lambda r);$$

therefore the additional part contributed to the left-hand side of 106 is

$$-n \sin 2n\pi \cos 2n\pi \int_0^\infty \int_0^a \frac{e^{-(\kappa+\lambda)z}}{r} J_n(\kappa r) J_n(\lambda r)\, dr dz.$$

Now this is symmetrical in κ and λ; therefore the same expression will occur on the right-hand side of 106, and consequently the formula 107 remains true for all real values of n. In the same way the formulæ 108 and 109 are true for all values of n.

To show the application of these results, we will employ the function

$$\phi = e^{-\lambda z} J_0(\lambda r)$$

to obtain the solution of a problem in the conduction of heat. Consider the solid cylinder bounded by the surfaces $r = 1$, $z = 0$, $z = +\infty$, and suppose that its convex surface is surrounded by a medium of temperature zero. Then when the flow of heat has become steady, the temperature V at any point in the cylinder must satisfy the equation

$$\nabla^2 V = 0$$

and moreover, when $r = 1$,

$$k \frac{\partial V}{\partial r} + hV = 0$$

where k is the conductivity of the material of the cylinder, and h is what Fourier calls the "external conductivity."

If we put $V = \phi$, the first condition is satisfied; and the second will also be satisfied, if

$$\lambda k J_0'(\lambda) + h J_0(\lambda) = 0. \qquad 110$$

Suppose, then, that λ is any root of this equation, and suppose, moreover, that the base of the cylinder is permanently heated so

that the temperature at a distance r from the centre is $J_0(\lambda r)$. Then the temperature at any point within the cylinder is

$$V = e^{-\lambda z} J_0(\lambda r),$$

because this satisfies all the conditions of the problem.

The equation 110 has an infinite number of real roots λ_1, λ_2, etc. so that we can construct a more general function

$$\phi = \sum_1^\infty A_s e^{-\lambda_s z} J_0(\lambda_s r) \qquad 111$$

and this will represent the temperature of the same cylinder when subject to the same conditions, except that the temperature at any point of the base is now given by

$$\phi_0 = \sum_1^\infty A_s J_0(\lambda_s r). \qquad 112$$

Now there does not appear to be any physical objection to supposing an arbitrary distribution of temperature over the base of the cylinder, provided the temperature varies continuously from point to point and is everywhere finite. In particular we may suppose the distribution symmetrical about the centre, and put

$$\phi_0 = f(r)$$

where $f(r)$ is any function of r which is one-valued, finite and continuous from $r=0$ to $r=1$. The question is whether this function can be reduced, for the range considered, to the form expressed by 112.

Assuming that this is so, we can at once obtain the coefficients A_s in the form of definite integrals; for if we put

$$f(r) = \Sigma A_s J_0(\lambda_s r) \qquad 113$$

it follows by 107, 109, and 110 that

$$\int_0^1 J_0(\lambda_s r) f(r)\, r\,dr = A_s \int_0^1 J_0^2(\lambda_s r)\, r\,dr$$
$$= \frac{A_s}{2}\left(\frac{h^2}{k^2\lambda_s^2} + 1\right) J_0^2(\lambda_s),$$

and therefore

$$A_s = \frac{2k^2\lambda_s^2}{(h^2 + k^2\lambda_s^2)\, J_0^2(\lambda_s)} \int_0^1 J_0(\lambda_s r) f(r)\, r\,dr. \qquad 114$$

Whenever the transformation 113 is legitimate, the function

$$\phi = \Sigma A_s e^{-\lambda_s z} J_0(\lambda_s r) \tag{115}$$

gives the temperature at any point of the cylinder, when its convex surface, as before, is surrounded by a medium of zero temperature, and the circular base is permanently heated according to the law

$$\phi_0 = f(r) = \Sigma A_s J_0(\lambda_s r);$$

the coefficients A_s being given by the formula 114.

A much more general form of potential function is obtained by putting

$$\phi = \Sigma (A \cos n\theta + B \sin n\theta) e^{-\lambda z} J_n(\lambda r) \tag{116}$$

where the summation refers to n and λ independently.

If we restrict the quantities n to integral values and take for the quantities λ the positive roots of $J_n(\lambda) = 0$, we have a potential which remains unaltered when θ is changed into $\theta + 2\pi$, which vanishes when $z = + \infty$, and also when $r = 1$. The value when $z = 0$ is

$$\phi_0 = \Sigma (A \cos n\theta + B \sin n\theta) J_n(\lambda r). \tag{117}$$

The function ϕ may be interpreted as the temperature at any point in a solid cylinder when the flow of heat is steady, the convex surface maintained at a constant temperature zero, and the base of the cylinder heated according to the law expressed by 117. We are led to inquire whether an arbitrary function $f(r, \theta)$, subject only to the conditions of being finite, one-valued, and continuous over the circle $r = 1$, can be reduced to the form of the right-hand member of 117.

Whenever this reduction is possible, it is easy to obtain the coefficients. Thus if, with a more complete notation, we have

$$f(r, \theta) = \Sigma\Sigma (A_{n,s} \cos n\theta + B_{n,s} \sin n\theta) J_n(\lambda_s r) \tag{118}$$

we find successively

$$\int_0^{2\pi} f(r, \theta) \cos n\theta \, d\theta = \sum_s \pi A_{n,s} J_n(\lambda_s r)$$

and

$$\int_0^1 \int_0^{2\pi} f(r, \theta) \cos n\theta J_n(\lambda_s r) \, r d\theta \, dr = \pi A_{n,s} \int_0^1 J_n^2(\lambda_s r) \, r dr$$

$$= \frac{\pi}{2} J_n'^2(\lambda_s) A_{n,s}$$

by 109; so that

$$A_{n,s} = \frac{2}{\pi J_n'^2(\lambda_s)} \int_0^1 \int_0^{2\pi} f(r,\theta) \cos n\theta J_n(\lambda_s r)\, r\, d\theta\, dr;$$

and in the same way (118′)

$$B_{n,s} = \frac{2}{\pi J_n'^2(\lambda_s)} \int_0^1 \int_0^{2\pi} f(r,\theta) \sin n\theta J_n(\lambda_s r)\, r\, d\theta\, dr.$$

Other physical problems may be constructed which suggest analytical expansions analogous to 113 and 118; some of these are given in the Examples.

We do not propose to discuss the validity of the expansions obtained in this chapter; to do so in a satisfactory way would involve a great many delicate considerations, and require a disproportionate amount of space. And after all, the value of these discussions to the practical physicist, in the present stage of applied mathematics, is not very great; for the difficulties of the analytical investigation are usually connected with the amount of restriction which must be applied to an arbitrary function in order that it may admit of expansion in the required way, and in the physical applications these restrictions are generally satisfied from the nature of the case. Such a work as Fourier's *Théorie Analytique de la Chaleur* is sufficient to show that the instinct of a competent physicist preserves him from mistakes in analysis, even when he employs functions of the most complicated and peculiar description.

For further information the reader is referred to Heine's *Kugelfunctionen*, 2nd edition, Vol. II. p. 210, and to a paper by the same author, entitled *Einige Anwendungen der Residuenrechnung von Cauchy*, Crelle, t. 89 (1880), pp. 19—39.*

As an example of the conclusions to which Dr Heine is led, consider the expression on the right-hand side of 113, where the coefficients A_s are determined by 114; then if the infinite series

$$\Sigma A_s J_0(\lambda_s r)$$

is uniformly convergent, its value is $f(r)$.

Similar considerations apply to the expansion 118, and others of the same kind. There are a few elementary arguments, which,

* See also the papers by du Bois-Reymond (*Math. Ann.*, iv. 362), Weber (ibid. vi. 146), and Hankel (ibid. viii. 471).

although not amounting to a demonstration, may help to explain the possibility of these expansions in some simple cases.

Suppose, for instance, that ϕ is a function which is one-valued, finite, and continuous all over the circle $r=1$, and admits of an absolutely convergent expansion

$$\phi = u_0 + u_1 + u_2 + \ldots = \sum_0^\infty u_s,$$

where u_s is a homogeneous rational integral function of degree s in Cartesian coordinates x, y. Then by putting $x = r\cos\theta$, $y = r\sin\theta$, this is transformed into

$$\phi = v_0 + v_1 r + v_2 r^2 + \ldots = \sum_0^\infty v_s r^s,$$

where v_s is an integral homogeneous function of $\cos\theta$ and $\sin\theta$.

By expressing v_s in terms of sines and cosines of multiples of θ, and rearranging the terms, ϕ may be reduced to the form

$$\begin{aligned}\phi &= \rho_0 + (\rho_1\cos\theta + \sigma_1\sin\theta) + (\rho_2\cos 2\theta + \sigma_2\sin 2\theta) + \ldots\\ &= \Sigma(\rho_s\cos s\theta + \sigma_s\sin s\theta),\end{aligned}$$

where ρ_s and σ_s are, generally speaking, infinite power-series in r, each beginning with a term in r^s.

Now if λ_1, λ_2, etc. are the roots of $J_s(\lambda)=0$, the series for $J_s(\lambda_1 r)$, $J_s(\lambda_2 r)$, etc. each begin with a term in r^s, and if we put

$$\rho_s = A_1^{(s)} J_s(\lambda_1 r) + A_2^{(s)} J_s(\lambda_2 r) + \ldots$$

it may be possible to determine the constants $A_1^{(s)}$, $A_2^{(s)}$, etc. so as to make the coefficients of the same power of r on both sides agree to any extent that may be desired. If the result of carrying out this comparison indefinitely leads to a convergent series

$$A_1^{(s)} J_s(\lambda_1 r) + A_2^{(s)} J_s(\lambda_2 r) + \ldots$$

we may legitimately write

$$\rho_s = \Sigma A_m^{(s)} J_s(\lambda_m r),$$

and in like manner we may arrive at a valid formula

$$\sigma_s = \Sigma B_m^{(s)} J_s(\lambda_m r);$$

and then

$$\phi = \sum_s\left[\cos s\theta \sum_m A_m^{(s)} J_s(\lambda_s r) + \sin s\theta \sum_m B_m^{(s)} J_s(\lambda_s r)\right],$$

which is equivalent to 118.

It will be understood that this does not in any way amount to a *proof* of the proposition: but it shows how, in a particular case of the kind considered, the expansion may be regarded as a straightforward algebraical transformation verifiable *a posteriori*.

CHAPTER VII.

COMPLEX THEORY.

MANY properties of the Bessel functions may be proved or illustrated by means of the theory of a complex variable, as explained, for instance, in Forsyth's *Theory of Functions*. A few of the most obvious of these applications will be given in the present chapter.

To avoid unnecessary complication, it will be supposed throughout that $n+\frac{1}{2}$ is real and positive, and, unless the contrary is expressly stated, that the real part of x is also positive.

Then, as already proved (p. 38),

$$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d\phi.$$

Now

$$\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d\phi = \int_{-1}^{+1}\cos(xt)(1-t^2)^{n-\frac{1}{2}}\,dt$$

$$= \int_{-1}^{+1} e^{xti}(1-t^2)^{n-\frac{1}{2}}\,dt$$

on the supposition that the integral is taken along the axis of real quantities from -1 to $+1$. We may therefore write

$$x^n\int_{-1}^{+1} e^{xzi}(1-z^2)^{n-\frac{1}{2}}\,dz = 2^n\sqrt{\pi}\,\Pi(n-\tfrac{1}{2})\,J_n(x), \qquad 119$$

and this will remain true for any finite path of integration from -1 to $+1$ which is reconcilable with the simple straight path.

We are thus naturally led to consider the function

$$u = x^n \int_\alpha^\beta e^{xzi} (1 - z^2)^{n-\frac{1}{2}} dz \tag{120}$$

where α, β are independent of x.

It will be found that

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(1 - \frac{n^2}{x^2}\right) u$$

$$= x^{n-1} \int_\alpha^\beta \{x(1 - z^2) + (2n + 1) iz\} e^{xzi} (1 - z^2)^{n-\frac{1}{2}} dz$$

$$= - ix^{n-1} \int_\alpha^\beta \frac{d}{dz} \{e^{xzi} (1 - z^2)^{n+\frac{1}{2}}\} dz;$$

therefore u will be a solution of Bessel's equation if

$$e^{xzi} (1 - z^2)^{n+\frac{1}{2}}$$

vanishes when $z = \alpha$ and also when $z = \beta$. Under the restrictions imposed upon x and n, the admissible values of α and β are $+1$, -1 and $k + \infty i$, where k is any finite real quantity. By assigning these special values to α and β and choosing different paths of integration, we obtain a large number of solutions in the form of definite integrals. Each solution must, of course, be expressible in the form $AJ_n + BJ_{-n}$ or $AJ_n + BY_n$; the determination of the constants is not always easy, and in fact this is the principal difficulty that has to be overcome.

It must be carefully borne in mind that, in calculating the value of u for any particular path, the function $(1 - z^2)^{n-\frac{1}{2}}$ must be taken to vary continuously. The value of u is not determinate until we fix the value of $(1 - z^2)^{n-\frac{1}{2}}$ at some one point of the path. If U is one value of u for a particular path, the general value for the same (or an equivalent) path is

$$u = e^{(2n-1)k\pi i}\, U,$$

where k is any real integer.

Consider the function

$$u_1 = x^n \int_1^{+\infty i} e^{xzi} (1 - z^2)^{n-\frac{1}{2}} dz; \tag{121}$$

we may choose as the path of integration a straight line from 1 to 0, followed by a straight line from 0 to ∞i. Then if t

denotes a real variable, and if we take that value of $(1-z^2)^{n-\frac{1}{2}}$ which reduces to $+1$ when $z=0$,

$$u_1 = x^n\left\{\int_1^0 e^{xti}(1-t^2)^{n-\frac{1}{2}}\,dt + i\int_0^\infty e^{-xt}(1+t^2)^{n-\frac{1}{2}}\,dt\right\}$$

$$= -x^n\int_0^{\frac{\pi}{2}}\cos(x\sin\phi)\cos^{2n}\phi\,d\phi$$

$$-\,ix^n\left\{\int_0^{\frac{\pi}{2}}\sin(x\sin\phi)\cos^{2n}\phi\,d\phi - \int_0^\infty e^{-x\sinh\phi}\cosh^{2n}\phi\,d\phi\right\}.$$

For convenience, let us write

$$C_n = 2^n\sqrt{\pi}\,\Pi\,(n-\tfrac{1}{2}); \qquad 122$$

then $$x^n\int_0^{\frac{\pi}{2}}\cos(x\sin\phi)\cos^{2n}\phi\,d\phi = \tfrac{1}{2}C_nJ_n,$$

and if we put

$$x^n\int_0^{\frac{\pi}{2}}\sin(x\sin\phi)\cos^{2n}\phi\,d\phi$$

$$-\,x^n\int_0^\infty e^{-x\sinh\phi}\cosh^{2n}\phi\,d\phi = \tfrac{1}{2}C_n\Upsilon_n, \qquad 123$$

Υ_n is also a solution of Bessel's equation, and one value of u_1 is

$$U_1 = -\tfrac{1}{2}C_n(J_n + i\Upsilon_n); \qquad 124$$

the general value being

$$u_1 = e^{(2n-1)k\pi i}\,U_1 \qquad 125$$

where k is any real integer.

In the same way, if

$$u_2 = x^n\int_{-1}^{+\infty i} e^{xzi}(1-z^2)^{n-\frac{1}{2}}\,dz, \qquad 126$$

taken along a straight line from -1 to 0, followed by a straight line from 0 to ∞i, one value of u_2 is

$$U_2 = \tfrac{1}{2}C_n(J_n - i\Upsilon_n) \qquad 127$$

and the general value is

$$u_2 = e^{(2n-1)k\pi i}\,U_2.$$

Now consider the integral

$$u_s = x^n \int_{+\infty i}^{+\infty i} e^{xzi} (1 - z^2)^{n-\frac{1}{2}}\, dz, \qquad 128$$

taken along a path inclosing the points -1 and $+1$, and such that throughout the integration $|z| > 1$, where $|z|$ means the absolute value, or modulus, of z.

Then, by the binomial theorem, we may put

$$(1 - z^2)^{n-\frac{1}{2}} = e^{(n-\frac{1}{2})\pi i} \{z^{2n-1} - (n - \tfrac{1}{2}) z^{2n-3} + \ldots\},$$

and therefore one value of u_s is given by

$$e^{-(n-\frac{1}{2})\pi i} u_s$$
$$= x^n \sum_{s=0}^{s=\infty} \frac{(-)^s (n - \frac{1}{2})(n - \frac{3}{2})\ldots(n - s + \frac{1}{2})}{s!} \int_{+\infty i}^{+\infty i} e^{xzi} z^{2n-2s-1}\, dz.$$

The integral on the right-hand side may be evaluated by taking the path along the axis of imaginary quantities from $+\infty i$ to ϵi, then round the origin, and then from ϵi to $+\infty i$. One value of the result is

$$i^{2n-2s} \int_\infty^0 e^{-xt} t^{2n-2s-1}\, dt + e^{4n\pi i}\, i^{2n-2s} \int_0^\infty e^{-xt} t^{2n-2s-1}\, dt,$$

which
$$= (-)^s e^{n\pi i} (e^{4n\pi i} - 1) \int_0^\infty e^{-xt} t^{2n-2s-1}\, dt$$
$$= (-)^s\, 2ie^{3n\pi i} \sin 2n\pi \frac{\Pi (2n - 2s - 1)}{x^{2n-2s}}.$$

Therefore one value of u_s is given by

$$\bar{u}_s = 2e^{4n\pi i} \sin 2n\pi \,\Sigma \frac{(n-\frac{1}{2})(n-\frac{3}{2})\ldots(n - s + \frac{1}{2})}{s!} \Pi (2n - 2s - 1)\, x^{-n+2s}.$$

Observing that

$$\frac{(n-\frac{1}{2})(n-\frac{3}{2})\ldots(n - s + \frac{1}{2})\, \Pi (2n - 2s - 1)}{\Pi s}$$
$$= \frac{1}{2^{2s}} \frac{\Pi (2n-1)\, \Pi (n - s - 1)}{\Pi (n-1)\, \Pi (s)},$$

and that

$$\Pi (n - s - 1)\, \Pi (s - n) = \pi \operatorname{cosec} (n - s)\, \pi = (-)^s \pi \operatorname{cosec} n\pi,$$

we have

$$\bar{u}_3 = 4\pi \cos n\pi \, e^{4n\pi i} \frac{\Pi(2n-1)}{\Pi(n-1)} \Sigma \frac{(-)^s x^{-n+2s}}{2^{2s}\Pi(s)\Pi(s-n)}$$

$$= \frac{2^{n+1}\pi^{\frac{3}{2}}}{\Pi(-n-\frac{1}{2})} e^{4n\pi i} J_{-n}(x)$$

on reduction.

The general value is $u_3 = e^{(2n-1)k\pi i}\bar{u}_3$; and by putting $k = -2$ we have the special value

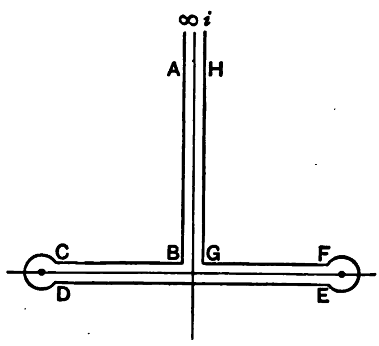
$$U_3 = \frac{2^{n+1}\pi^{\frac{3}{2}}}{\Pi(-n-\frac{1}{2})} J_{-n}(x)$$

$$= 2C_n \cos n\pi \, J_{-n}(x), \qquad 129$$

and the general value

$$u_3 = e^{(2n-1)k\pi i} U_3.$$

It has been convenient to suppose that $|z| > 1$ throughout the integration; but the value of u_3 will not be altered if we suppose the integration taken along the path $ABCDEFGH$ represented in the figure below.



If the argument of $(1-z^2)^{n-\frac{1}{2}}$ is taken to be zero along DE, we must take it to be $-(2n-1)\pi$ along AB, BC, and $+(2n-1)\pi$ along FG, GH. Hence we have one value of u_3 given by

$$u_3 = -e^{-(2n-1)\pi i} U_2 + e^{(2n-1)\pi i} U_1 + x^n \int_{-1}^{+1} e^{xti}(1-t^2)^{n-\frac{1}{2}} dt$$

$$= C_n\left[\tfrac{1}{2}e^{-2n\pi i}(J_n - i\Upsilon_n) + \tfrac{1}{2}e^{2n\pi i}(J_n + i\Upsilon_n) + J_n\right]$$

$$= 2C_n \cos n\pi\,(J_n \cos n\pi - \Upsilon_n \sin n\pi). \qquad 130$$

Since n is real, this must agree with the real value of u_3, that is, with $2C_n \cos n\pi\, J_{-n}$. Hence we infer that

$$J_{-n} = J_n \cos n\pi - \Upsilon_n \sin n\pi. \tag{131}$$

The formula 123, by which Υ_n was defined, is intelligible only so long as the real part of x is positive: we may, however, if we like, regard 131 as a definition of Υ_n for all values of x. Then 123 still remains valid so long as the integrals are finite.

When n is an integer, 131 becomes an identity, but the value of Υ_n is not really indeterminate. To find its real value, write $n - \epsilon$ for n, ϵ being a small positive quantity, and n being supposed an integer; then

$$J_{-(n-\epsilon)} - (-)^n \cos \epsilon\pi J_{n-\epsilon} = (-)^n \sin \epsilon\pi \Upsilon_{n-\epsilon}.$$

Divide both sides by $(-)^n \epsilon$ and make ϵ vanish; thus

$$\pi\Upsilon_n = \underset{\epsilon=0}{Lt} \frac{(-)^n J_{-(n-\epsilon)} - J_{(n-\epsilon)}}{\epsilon}.$$

The expression on the right-hand side may be written in the form

$$\left(\frac{x}{2}\right)^{-n+\epsilon} \sum_0^{n-1} \frac{(-)^{n+s}}{\Pi(s)} \frac{1}{\epsilon\Pi(-n+\epsilon+s)} \left(\frac{x}{2}\right)^{2s} + \left(\frac{x}{2}\right)^n \sum_0^\infty (-)^s \frac{f(\epsilon)}{\epsilon} \left(\frac{x}{2}\right)^{2s},$$

where

$$f(\epsilon) = \frac{1}{\Pi(n+s)\,\Pi(s+\epsilon)} \left(\frac{x}{2}\right)^\epsilon - \frac{1}{\Pi(s)\,\Pi(n+s-\epsilon)} \left(\frac{x}{2}\right)^{-\epsilon}.$$

Let us use the notation

$$\psi(x) = \frac{d}{dx} \log \Pi(x) = \frac{\Pi'(x)}{\Pi(x)}; \tag{132}$$

then the limit of $f(\epsilon)/\epsilon$ when ϵ is zero is

$$\frac{1}{\Pi(s)\,\Pi(n+s)} \left\{2 \log \frac{x}{2} - \psi(s) - \psi(n+s)\right\}.$$

Again (p. 12),

$$\Pi(-n+\epsilon+s)\,\Pi(n-\epsilon-s-1)=\pi\operatorname{cosec}(n-\epsilon-s)\,\pi$$
$$=\frac{(-)^{n-s-1}\pi}{\sin\epsilon\pi};$$

therefore

$$\operatorname*{Lt}_{\epsilon=0}\epsilon\Pi(-n+\epsilon+s)=\frac{(-)^{n-s-1}}{\Pi(n-s-1)}.$$

Finally, then, when n is a positive integer,

$$\pi\Upsilon_n=-\left(\frac{x}{2}\right)^{-n}\sum_0^{n-1}\frac{\Pi(n-s-1)}{\Pi(s)}\left(\frac{x}{2}\right)^{2s}$$
$$+\left(\frac{x}{2}\right)^n\sum_0^\infty\frac{(-)^s}{\Pi(s)\,\Pi(n+s)}\left(\frac{x}{2}\right)^{2s}\left\{2\log\frac{x}{2}-\psi(s)-\psi(n+s)\right\}. \quad 133$$

Comparing this with the formula 30 (p. 14) we see that, when n is a positive integer,

$$\pi\Upsilon_n=2\{Y_n-(\log 2+\psi(0))\,J_n\}, \quad 134$$

or, with the notation of p. 40, and on substituting for Υ_n its value as a definite integral,

$$Y_n=(\log 2-\gamma)\,J_n+\frac{\pi}{2}\Upsilon_n$$
$$=\frac{x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}\left\{(\log 2-\gamma)\int_0^\pi\cos(x\cos\phi)\sin^{2n}\phi\,d\phi\right.$$
$$+\pi\int_0^{\frac{\pi}{2}}\sin(x\sin\phi)\cos^{2n}\phi\,d\phi$$
$$\left.-\pi\int_0^\infty e^{-x\sinh\phi}\cosh^{2n}\phi\,d\phi\right\}. \quad 135$$

When x is a very large positive quantity, the last integral on the right-hand side may be neglected; and then, by applying Stokes's method* to the other two integrals, we obtain the approximate value of Y_n given on p. 40 above.

There would be a certain advantage in taking J_n and Υ_n as the fundamental solutions of Bessel's equation in all cases when n is positive. This course is practically adopted by Hankel, whose memoir in the first volume of the *Mathematische Annalen* we have been following in this chapter. Hankel writes Y_n for a

* Or, preferably, that of Lipschitz, explained later on, pp. 69—71.

function we may denote, for the moment, by $\overline{Y}_n$; this is defined for all values of n by the formula

$$\left.\begin{aligned}\overline{Y}_n &= 2\pi e^{n\pi i}\frac{J_n \cos n\pi - J_{-n}}{\sin 2n\pi}\\ &= \frac{\pi e^{n\pi i}}{\cos n\pi}\Upsilon_n\end{aligned}\right\} \qquad 136$$

(see 131 above).

Since, however, Neumann's notation is now being generally adopted, we shall continue to use Y_n in the sense previously defined. It may be remarked that, when n is a positive integer,

$$\overline{Y}_n = \pi\Upsilon_n = 2\{Y_n - (\log 2 - \gamma) J_n\}. \qquad 137$$

Hitherto the real part of x has been supposed positive; we will now suppose that x is a pure imaginary. If in Bessel's equation we put $x = ti$, it becomes

$$\frac{d^2u}{dt^2} + \frac{1}{t}\frac{du}{dt} - \left(1 + \frac{n^2}{t^2}\right)u = 0. \qquad 138$$

By proceeding as in Chap. II. it is easily found that, when n is not an integer, the equation has two independent solutions $I_n(t)$, $I_{-n}(t)$, defined by

$$\left.\begin{aligned}I_n(t) &= i^{-n} J_n(it) = \sum_0^\infty \frac{t^{n+2s}}{2^{n+2s}\Pi(s)\Pi(n+s)}\\ I_{-n}(t) &= i^n J_{-n}(it) = \sum_0^\infty \frac{t^{-n+2s}}{2^{-n+2s}\Pi(s)\Pi(-n+s)}\end{aligned}\right\} \qquad 139$$

one of which vanishes when $t = 0$, while the other becomes infinite.

When n is an integer we have the solution I_n as before, and there will be another solution obtained by taking the real part of $i^{-n}Y_n(it)$. But it is found that *both* these functions (and in like manner I_n and I_{-n}) become infinite when $t = \infty$, and it is important for certain applications to discover a solution which vanishes when t is infinite. We shall effect this by returning to the equation 123, by which Υ_n was originally defined. If in the expression on the left-hand side of that equation we put $x = ti$, it becomes

$$i^n t^n \{- U_n + iV_n\},$$

where $U_n = \int_0^\infty \cos(t\sinh\phi)\cosh^{2n}\phi\,d\phi$,

$$V_n = \int_0^{\frac{\pi}{2}} \sinh(t\sin\phi)\cos^{2n}\phi\,d\phi + \int_0^\infty \sin(t\sinh\phi)\cosh^{2n}\phi\,d\phi.$$

The function V_n is infinite whether n is positive or negative; but if $2n+1$ is negative U_n is finite, and it may be verified that

$$t^n U_n = t^n \int_0^\infty \cos(t\sinh\phi)\cosh^{2n}\phi\,d\phi$$

is a solution of Bessel's transformed equation 138. Now although this does not give us what we want, namely a solution when n is positive which shall vanish when t is infinite, it suggests that we should try the function

$$t^{-n}\int_0^\infty \frac{\cos(t\sinh\phi)}{\cosh^{2n}\phi}\,d\phi$$

which is obtained from $t^n U_n$ by changing n into $-n$. This function obviously vanishes when $t=\infty$, and it is easily verified that it satisfies the equation 138, so that it is the solution we require.

It will be found convenient to write

$$K_n(t) = \frac{2^n\sqrt{\pi}t^{-n}}{\Pi(-\frac{1}{2}-n)}\int_0^\infty \frac{\cos(t\sinh\phi)\,d\phi}{\cosh^{2n}\phi}, \qquad 140$$

reducing, when n is an integer, to

$$K_n = (-)^n 1.3.5\ldots(2n-1)\,t^{-n}\int_0^\infty \frac{\cos(t\sinh\phi)\,d\phi}{\cosh^{2n}\phi}.$$

Then I_n and K_n are solutions of 138 which are available for all positive values of n, and which are the most convenient solutions to take as the fundamental ones, whether or not n is an integer.

As in Chapter II. it may be proved that

$$\left.\begin{aligned} I'_n &= \frac{n}{t}I_n + I_{n+1} \\ I'_n &= I_{n-1} - \frac{n}{t}I_n \\ I_{n+1} + \frac{2n}{t}I_n - I_{n-1} &= 0 \\ I'_0 &= I_1 \end{aligned}\right\} \qquad 141$$

and that the functions K_n are connected by relations of precisely the same form. It is in order to preserve this analogy that it has been thought desirable to modify the notation proposed in Basset's *Hydrodynamics*, vol. II. p. 19. The function here called $K_n(t)$ is 2^n times Mr. Basset's K_n.

By applying Stokes's method to the differential equation 138 we obtain the semiconvergent expansion

$$u = Ae^t t^{-\frac{1}{2}}\left\{1 - \frac{(4n^2-1)}{8t} + \frac{(4n^2-1)(4n^2-9)}{2!(8t)^2} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8t)^3} + \dots\right\}$$
$$+ Be^{-t} t^{-\frac{1}{2}}\left\{1 + \frac{(4n^2-1)}{8t} + \frac{(4n^2-1)(4n^2-9)}{2!(8t)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8t)^3} + \dots\right\}$$

which terminates, and gives an exact solution, when n is half an odd integer. This may be used for approximate calculation when t is large.

When $u = K_n$, it is evident that $A = 0$. A somewhat troublesome and not very satisfactory process, suggested by a formula in Laurent's *Traité d'Analyse*, t. III. p. 255, leads to the conclusion that $B = \sqrt{\frac{1}{2}\pi}\,.\cos n\pi$, so that the semiconvergent expression for K_n is

$$K_n = \sqrt{\frac{\pi}{2t}}\cos n\pi\,.\,e^{-t}\left\{1 + \frac{(4n^2-1)}{8t} + \frac{(4n^2-1)(4n^2-9)}{2!(8t)^2} + \dots\right\}. \quad 142$$

The corresponding expression for I_n is

$$I_n = \sqrt{\frac{1}{2\pi t}}\,e^t\left\{1 - \frac{(4n^2-1)}{8t} + \frac{(4n^2-1)(4n^2-9)}{2!(8t)^2} - \dots\right\}. \quad 143$$

There does not appear to be much reason to doubt the correctness of these results, since the formula for I_n works out very fairly, even for comparatively small values of t, and the formulæ are consistent, as they should be, with

$$I_{n+1}K_n - I_nK_{n+1} = \frac{1}{t}\cos n\pi. \quad 144$$

The formula 30 (p. 14) leads us to conclude that if $R_n(t)$ is the real part of $i^{-n}\,Y_n(it)$, $K_n(t)$ must agree, up to a numerical factor, with

$$R_n(t) - (\log 2 - \gamma)\,I_n(t);$$

this is confirmed by Stokes's investigation, *Camb. Phil. Trans.*, vol. IX. p. [38].

It would be easy to multiply these applications of the complex theory to any extent; we will conclude this chapter by giving an alternative proof of the semiconvergent expansion of J_n obtained in Chap. IV. above. The method is that of Lipschitz (*Crelle* LVI., p. 189), who works out in detail the case when $n=0$, and indicates how the general case may be treated.

Let us suppose that $n+\frac{1}{2}$ and the real part of x are both positive. Consider the integral

$$u=\int e^{xzi}(1-z^2)^{n-\frac{1}{2}}\,dz$$

taken, in the positive direction, along the contour of the rectangle whose vertices are at the points corresponding to the quantities -1, $+1$, $1+hi$, $-1+hi$, where h is real and positive. Then the total value of the integral is zero, and by expressing it as the sum of four parts arising from the sides of the rectangle, we obtain the relation

$$\begin{aligned}0=\int_{-1}^{+1}&e^{xti}(1-t^2)^{n-\frac{1}{2}}\,dt+i\int_0^h e^{x(1+it)i}\{1-(1+it)^2\}^{n-\frac{1}{2}}\,dt\\&-\int_{-1}^{+1}e^{x(hi+t)i}\{1-(hi+t)^2\}^{n-\frac{1}{2}}\,dt\\&-i\int_0^h e^{x(-1+it)i}\{1-(-1+it)^2\}^{n-\frac{1}{2}}\,dt,\end{aligned}$$

where t is, throughout, a real variable.

The first integral is $C_n x^{-n}J_n$, and when $h=\infty$, the third integral vanishes: hence

$$\begin{aligned}C_n x^{-n}J_n(x)=ie^{-ix}&\int_0^\infty(2it+t^2)^{n-\frac{1}{2}}e^{-xt}\,dt\\&-ie^{ix}\int_0^\infty(-2it+t^2)^{n-\frac{1}{2}}e^{-xt}\,dt.\end{aligned}\qquad 145$$

The argument of $2it+t^2$ must be taken to be $\frac{\pi}{2}$ when t vanishes: hence

$$(2it+t^2)^{n-\frac{1}{2}}=2^{n-\frac{1}{2}}e^{\frac{1}{4}(2n-1)\pi i}t^{n-\frac{1}{2}}\left(1+\frac{t}{2i}\right)^{n-\frac{1}{2}},$$

where $t^{n-\frac{1}{2}}$ is real, and the argument of $1+\frac{t}{2i}$ vanishes when $t=0$. Applying a similar transformation to $(-2it+t^2)^{n-\frac{1}{2}}$, and putting

$$\frac{(2n+1)\pi}{4}-x=\psi,$$

we have

$$2^{-n+\frac{1}{2}}C_n x^{-n} J_n(x)=e^{\psi i}\int_0^\infty e^{-xt}\,t^{n-\frac{1}{2}}\left(1+\frac{t}{2i}\right)^{n-\frac{1}{2}}dt$$
$$+e^{-\psi i}\int_0^\infty e^{-xt}\,t^{n-\frac{1}{2}}\left(1-\frac{t}{2i}\right)^{n-\frac{1}{2}}dt.$$

Now put $xt=\xi$; then

$$2^{-n+\frac{1}{2}}C_n x^{\frac{1}{2}} J_n(x)=e^{\psi i}\int_0^\infty e^{-\xi}\,\xi^{n-\frac{1}{2}}\left(1+\frac{\xi}{2ix}\right)^{n-\frac{1}{2}}d\xi$$
$$+e^{-\psi i}\int_0^\infty e^{-\xi}\,\xi^{n-\frac{1}{2}}\left(1-\frac{\xi}{2ix}\right)^{n-\frac{1}{2}}d\xi.$$

By Maclaurin's theorem, we may write

$$\left(1+\frac{\xi}{2ix}\right)^{n-\frac{1}{2}}=1+\frac{(n-\frac{1}{2})\,\xi}{2ix}+\frac{(n-\frac{1}{2})(n-\frac{3}{2})\,\xi^2}{2\,!\,(2ix)^2}+\dots$$
$$+\frac{(n-\frac{1}{2})(n-\frac{3}{2})\dots(n-s+\frac{3}{2})}{(s-1)\,!}\left(\frac{\xi}{2ix}\right)^{s-1}$$
$$+\frac{(n-\frac{1}{2})(n-\frac{3}{2})\dots(n-s+\frac{1}{2})}{s\,!}\left(\frac{\xi}{2ix}\right)^{s}\left(1+\frac{\theta\xi}{2ix}\right)^{n-s-\frac{1}{2}},$$

where θ is some proper fraction, and $\left(1-\frac{\xi}{2ix}\right)^{n-\frac{1}{2}}$ may be expanded in a similar way. Substitute these expressions in the preceding equation, and make use of the formula

$$\int_0^\infty e^{-\xi}\,\xi^{n-\frac{1}{2}+s}\,d\xi=\Pi\left(n-\tfrac{1}{2}+s\right);$$

then, if we further observe that

$$C_n=2^n\sqrt{\pi}\,\Pi\left(n-\tfrac{1}{2}\right),$$

we obtain

$$\sqrt{\frac{\pi x}{2}}\,.\,J_n(x)=\cos\psi+\frac{n^2-\frac{1}{4}}{2x}\sin\psi-\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{2\,!\,(2x)^2}\cos\psi+\dots$$

(to s terms)

$$+R, \qquad 146$$

where

$$2\Pi(n-\tfrac{1}{2})R = \frac{(n-\frac{1}{2})(n-\frac{3}{2})\dots(n-s+\frac{1}{2})}{s!}$$

$$\left[e^{\psi i}\int_0^\infty \left(\frac{\xi}{2ix}\right)^s \left(1+\frac{\theta\xi}{2ix}\right)^{n-s-\frac{1}{2}} \xi^{n-\frac{1}{2}} e^{-\xi}\, d\xi\right.$$

$$\left.+\,e^{-\psi i}\int_0^\infty \left(-\frac{\xi}{2ix}\right)^s \left(1-\frac{\theta'\xi}{2ix}\right)^{n-s-\frac{1}{2}} \xi^{n-\frac{1}{2}} e^{-\xi}\, d\xi\right].$$

Each of the integrals on the right hand is increased in absolute value if the quantities under the integral sign are replaced by their moduli: moreover, when $s > n - \frac{1}{2}$, the values are still further increased by replacing

$$\text{mod}\left(1+\frac{\theta\xi}{2ix}\right),\quad \text{mod}\left(1-\frac{\theta'\xi}{2ix}\right)$$

each by unity. *A fortiori*, if we put $|x| = a$,

$$|R| < \frac{(n-\frac{1}{2})(n-\frac{3}{2})\dots(n-s+\frac{1}{2})}{\Pi s\,\Pi(n-\frac{1}{2})}\left(\frac{1}{2a}\right)^s \int_0^\infty e^{-\xi}\xi^{n+s-\frac{1}{2}}\, d\xi$$

$$< \frac{(n-\frac{1}{2})(n-\frac{3}{2})\dots(n-s+\frac{1}{2})}{\Pi s\,\Pi(n-\frac{1}{2})} \cdot \frac{\Pi(n+s-\frac{1}{2})}{(2a)^s}.$$

The expression on the right hand is precisely the absolute value of the coefficient of $\cos\psi$ or $\sin\psi$, as the case may be, in the $(s+1)$th term of the series

$$\cos\psi + \frac{(n^2-\frac{1}{4})}{2x}\sin\psi - \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{2!\,(2x)^2}\cos\psi + \dots$$

so that the semiconvergence of this expansion, and its approximate representation of $\sqrt{\frac{\pi x}{2}}\,.\,J_n(x)$ are fully established.

CHAPTER VIII.

DEFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS.

MANY definite integrals involving Bessel functions have been evaluated by different mathematicians, more especially by Weber, Sonine, Hankel, and Gegenbauer. In the present chapter we shall give a selection of these integrals, arranged in as natural an order as the circumstances seem to admit. Others will be found in the examples at the end of the book.

By the formula 42 (p. 18) or 90 (p. 38) we have

$$J_0(bx) = \frac{1}{\pi}\int_0^\pi \cos(bx\cos\phi)\,d\phi.$$

Suppose that b is real, and let a be a real positive quantity: then

$$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\pi}\int_0^\infty dx \int_0^\pi e^{-ax}\cos(bx\cos\phi)\,d\phi.$$

Under the conditions imposed upon a and b, we may change the order of integration on the right hand, and make use of the formula

$$\int_0^\infty e^{-mx}\cos nx\,dx = \frac{m}{m^2+n^2};$$

thus

$$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\pi}\int_0^\pi \frac{a\,d\phi}{a^2+b^2\cos^2\phi}$$
$$= \frac{1}{\sqrt{a^2+b^2}}.$$

This result will still be true for complex values of a and b provided that the integral is convergent. Now if

$$a = a_1 + a_2 i,\quad b = b_1 + b_2 i,$$

the expression $e^{-ax} J_0(bx)$, when x is very large, behaves like

$$\frac{e^{-a_1 x}\cosh b_2 x}{\sqrt{x}} P(x),$$

where $P(x)$ is a trigonometrical function of x; hence the integral is convergent if

$$a_1 \geqq |b_2|$$

and under this condition we shall still have

$$\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}, \qquad 147$$

that value of $\sqrt{a^2+b^2}$ being taken which reduces to a when $b=0$.

When b is real and positive, we may put $a=0$, and thus obtain

$$\int_0^\infty J_0(bx)\,dx = \frac{1}{b}, \qquad 148$$

and as a special case

$$\int_0^\infty J_0(x)\,dx = 1. \qquad 149$$

Again let b be real and positive, and put ai instead of a, a being real and positive; thus

$$\int_0^\infty e^{-axi} J_0(bx)\,dx = \frac{1}{\sqrt{b^2-a^2}}.$$

If $b^2 > a^2$ the positive value of $\sqrt{b^2-a^2}$ must be taken; if $b^2 < a^2$ we must put

$$\sqrt{b^2-a^2} = +i\sqrt{a^2-b^2}.$$

We thus obtain Weber's results

$$\left.\begin{aligned}\int_0^\infty J_0(bx)\cos ax\,dx &= \frac{1}{\sqrt{b^2-a^2}}\\ \int_0^\infty J_0(bx)\sin ax\,dx &= 0\end{aligned}\right\} b^2 > a^2. \qquad 150$$

$$\left.\begin{aligned}\int_0^\infty J_0(bx)\cos ax\,dx &= 0\\ \int_0^\infty J_0(bx)\sin ax\,dx &= \frac{1}{\sqrt{a^2-b^2}}\end{aligned}\right\} a^2 > b^2. \qquad 151$$

When $a^2 = b^2$ the integrals become divergent; the reason is that they ultimately behave like

$$\int_0^\infty \frac{dx}{\sqrt{x}}$$

instead of like
$$\int_0^\infty \frac{\cos(a-x)\,dx}{\sqrt{x}}.$$

The formula 147 with which we started is due to Lipschitz (*Crelle* LVI.); another due to the same author is

$$\int_0^\infty x^{m-1} J_0(ax)\,dx = \frac{\Pi(m-1)\,\Pi\left(-\frac{m+1}{2}\right)\Pi\left(\frac{m}{2}-1\right)}{2\pi^{\frac{3}{2}} a^m}\sin m\pi, \qquad 152$$

where m is a positive proper fraction.

To prove this we observe that

$$\int_0^\infty x^{m-1} J_0(ax)\,dx = \frac{2}{\pi}\int_0^\infty dx \int_0^{\frac{\pi}{2}} x^{m-1}\cos(ax\cos\phi)\,d\phi,$$

and on changing the order of integration this becomes

$$\frac{2\Pi(m-1)\cos\frac{m\pi}{2}}{\pi a^m}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\cos^m\phi}$$

which
$$= \frac{\Pi(m-1)\cos\frac{m\pi}{2}}{\pi a^m}\int_0^1 t^{-\frac{m+1}{2}}(1-t)^{-\frac{1}{2}}\,dt$$

$$= \frac{\Pi(m-1)\cos\frac{m\pi}{2}}{\pi a^m}\cdot\frac{\Pi\left(-\frac{m+1}{2}\right)\Pi\left(-\frac{1}{2}\right)}{\Pi\left(-\frac{m}{2}\right)}.$$

Now
$$\Pi\left(-\frac{m}{2}\right)\Pi\left(\frac{m}{2}-1\right) = \pi\operatorname{cosec}\frac{m\pi}{2}$$

and
$$\Pi\left(-\frac{1}{2}\right) = \sqrt{\pi}$$

(see p. 12); thus the expression reduces to

$$\frac{\Pi(m-1)\cos\frac{m\pi}{2}}{\pi a^m}\cdot\frac{\sqrt{\pi}\,\Pi\left(-\frac{m+1}{2}\right)\Pi\left(\frac{m}{2}-1\right)\sin\frac{m\pi}{2}}{\pi};$$

that is, to

$$\frac{\Pi(m-1)\,\Pi\left(-\frac{m+1}{2}\right)\Pi\left(\frac{m}{2}-1\right)}{2\pi^{\frac{1}{2}}a^m}\sin m\pi,$$

the value given above.

The result may be written in various other forms: thus if we make use of the formula

$$\Pi(m-1)\,\Pi\left(-\frac{1+m}{2}\right)\cos\frac{m\pi}{2}=2^{m-1}\sqrt{\pi}\,\Pi\left(\frac{m}{2}-1\right)$$

it becomes

$$\left.\begin{aligned}\int_0^\infty x^{m-1}J_0(ax)\,dx&=\frac{2^{m-1}\left\{\Pi\left(\frac{m}{2}-1\right)\right\}^2}{\pi a^m}\sin\frac{m\pi}{2}\\ &=\frac{2^{m-1}\,\Pi\left(\frac{m}{2}-1\right)}{a^m\,\Pi\left(-\frac{m}{2}\right)}\end{aligned}\right\} \qquad 153$$

reducing, as might have been expected, to $1/a$ when $m=1$.

We will next consider a group of integrals which have been obtained by Weber (*Crelle* LXIX.) by means of a very ingenious analysis.

Let V be a function which is one-valued, finite and continuous, as well as its space-flux in any direction, throughout the whole of space, and which also satisfies the equation

$$\nabla^2V+m^2V=0.$$

Using polar coordinates r, θ, ϕ, and putting $\cos\theta=\mu$, this equation is

$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial V}{\partial\mu}\right)+\frac{1}{1-\mu^2}\frac{\partial^2V}{\partial\phi^2}\right\}=-m^2V.$$

Let $\quad \omega=\int_{-1}^{+1}\int_{-\pi}^{+\pi}Vd\mu\,d\phi$; then, observing that

$$\int_{-1}^{+1}\int_{-\pi}^{+\pi}d\mu\,d\phi\left\{\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial V}{\partial\mu}\right)+\frac{1}{1-\mu^2}\frac{\partial^2V}{\partial\phi^2}\right\}=0$$

because $\frac{\partial V}{\partial \phi}$ and $\frac{\partial V}{\partial \mu}$ are one-valued, we have

$$-m^2\omega = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \omega}{\partial r}\right)$$

or, which is the same thing,

$$\frac{\partial^2}{\partial r^2}(r\omega) + m^2 r\omega = 0;$$

whence $$\omega = \frac{1}{r}(A \sin mr + B \cos mr),$$

where A and B are independent of r. If ω is finite when $r=0$ we must have $B=0$, and

$$\omega = \frac{\omega_0 \sin mr}{mr},$$

where ω_0 is the value of ω when $r=0$. Now from the definition of ω it is clear that, if V_0 is the value of V when $r=0$,

$$\omega_0 = V_0 \int_{-1}^{+1}\int_{-\pi}^{+\pi} d\mu\, d\phi = 4\pi V_0$$

and therefore $$\omega = \frac{4\pi V_0}{m}\frac{\sin mr}{r}. \qquad 154$$

Now consider V as a function of rectangular coordinates a, b, c and put

$$V = \Phi(a, b, c);$$

moreover let us write

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(a, b, c)\, e^{-p^2\{(a-x)^2+(b-y)^2+(c-z)^2\}}\, da\, db\, dc = \Omega. \qquad 155$$

Then if we introduce polar coordinates by writing

$$a - x = r \sin\theta \cos\phi,$$
$$b - y = r \sin\theta \sin\phi,$$
$$c - z = r\cos\theta, \qquad \cos\theta = \mu,$$

we have $$\Omega = \int_0^{\infty} r^2 e^{-p^2 r^2}\, dr \int_{-1}^{+1}\int_{-\pi}^{+\pi} \Phi' d\mu\, d\phi,$$

where Φ' is the transformed expression for Φ.

By the preceding theorem this is

$$\Omega = \frac{4\pi}{m} \Phi_0' \int_0^\infty r e^{-p^2 r^2} \sin mr \, dr$$

$$= \frac{4\pi}{m} \Phi(x, y, z) \cdot \frac{m}{4p^3} \sqrt{\pi}\, e^{-\frac{m^2}{4p^2}}.$$

Hence, finally,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(a, b, c)\, e^{-p^2\{(a-x)^2 + (b-y)^2 + (c-z)^2\}}\, da\, db\, dc$$

$$= \frac{\pi^{\frac{3}{2}}}{p^3} e^{-\frac{m^2}{4p^2}} \Phi(x, y, z). \qquad 156$$

Suppose now that Φ is independent of c; then since

$$\int_{-\infty}^{+\infty} e^{-p^2 (c-z)^2} dc = \int_{-\infty}^{+\infty} e^{-p^2 t^2} dt = \sqrt{\pi}/p$$

we have

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(a, b)\, e^{-p^2\{(a-x)^2 + (b-y)^2\}}\, da\, db = \frac{\pi}{p^2} e^{-\frac{m^2}{4p^2}} \Phi(x, y), \qquad 157$$

the equation satisfied by Φ being

$$\frac{\partial^2 \Phi}{\partial a^2} + \frac{\partial^2 \Phi}{\partial b^2} + m^2 \Phi = 0.$$

In particular we may put

$$\Phi = J_0(mr),$$

where $\qquad r^2 = a^2 + b^2, \quad a = r\cos\theta, \quad b = r\sin\theta,$

and suppose that $\quad x = y = 0$;

then the formula becomes, after integrating with respect to θ,

$$\int_0^\infty r e^{-p^2 r^2} J_0(mr)\, dr = \frac{1}{2p^2} e^{-\frac{m^2}{4p^2}}. \qquad 158$$

More generally, by putting

$$\Phi = J_n(mr)(A \cos n\theta + B \sin n\theta),$$

$$x = \rho \cos \beta, \qquad y = \rho \sin \beta,$$

we obtain

$$e^{-p^2\rho^2} \int_0^\infty r e^{-p^2 r^2} J_n(mr)\, dr \int_{-\pi}^{+\pi} e^{2p^2 \rho r \cos(\theta - \beta)} (A \cos n\theta + B \sin n\theta)\, d\theta$$

$$= \frac{\pi}{p^2} e^{-\frac{m^2}{4p^2}} J_n(m\rho)(A \cos n\beta + B \sin n\beta). \qquad 159$$

In this formula put

$$A=1,\quad B=i,\quad \beta=\tfrac{1}{2}\pi;$$

then the integral with respect to θ is

$$\int_{-\pi}^{+\pi} e^{2p^2\rho r\sin\theta+n\theta i}\,d\theta$$

$$=2\int_0^{\pi}\cos(2ip^2\rho r\sin\theta-n\theta)\,d\theta$$

$$=2\pi i^n I_n(2p^2\rho r).$$

The formula thus becomes, after substitution, and division of both sides by $2\pi i^n e^{-p^2\rho^2}$,

$$\int_0^\infty re^{-p^2r^2}J_n(mr)\,I_n(2p^2\rho r)\,dr=\frac{1}{2p^2}e^{-\frac{m^2}{4p^2}+p^2\rho^2}J_n(m\rho),$$

or, more symmetrically, putting λ for m, and μ for $2p^2\rho$,

$$\int_0^\infty re^{-p^2r^2}J_n(\lambda r)\,I_n(\mu r)\,dr=\frac{1}{2p^2}e^{-\frac{\lambda^2-\mu^2}{4p^2}}J_n\left(\frac{\lambda\mu}{2p^2}\right), \qquad 160$$

or again, changing μ into $i\mu$, which does not affect the convergence of the integral,

$$\int_0^\infty re^{-p^2r^2}J_n(\lambda r)\,J_n(\mu r)\,dr=\frac{1}{2p^2}e^{-\frac{\lambda^2+\mu^2}{4p^2}}I_n\left(\frac{\lambda\mu}{2p^2}\right). \qquad 161$$

By making μ infinitesimal, we obtain the additional result

$$\int_0^\infty r^{n+1}e^{-p^2r^2}J_n(\lambda r)\,dr=\frac{\lambda^n}{(2p^2)^{n+1}}e^{-\frac{\lambda^2}{4p^2}}. \qquad 162$$

In all these formulæ the real part of p^2 must be positive in order to secure the convergence of the integrals.

After this singularly beautiful analysis, Weber proceeds to evaluate the integral

$$\int_0^\infty r^{q-n-1}J_n(\lambda r)\,dr$$

which is convergent so long as

$$0<q<n+\tfrac{3}{2}.$$

It is known that

$$\frac{1}{r^{2n+2-q}}=\frac{1}{\Pi(n-\frac{1}{2}q)}\int_0^\infty x^{n-\frac{1}{2}q}e^{-r^2x}\,dx;$$

therefore

$$\int_0^\infty r^{q-n-1} J_n(\lambda r)\, dr = \frac{1}{\Pi(n-\frac{1}{2}q)} \int_0^\infty dr \int_0^\infty x^{n-\frac{1}{2}q} r^{n+1} J_n(\lambda r)\, e^{-r^2 x}\, dx.$$

Now it may be shown (*l. c.* p. 230) that the value of the double integral is not affected by changing the order of integration; if we do this, and make use of 162, we obtain

$$\int_0^\infty r^{q-n-1} J_n(\lambda r)\, dr = \frac{\lambda^n}{2^{n+1}\,\Pi(n-\frac{1}{2}q)} \int_0^\infty e^{-\frac{\lambda^2}{4x}} x^{-\frac{1}{2}q-1}\, dx$$

$$= \frac{\lambda^n}{2^{n+1}\,\Pi(n-\frac{1}{2}q)} \int_0^\infty e^{-\frac{1}{4}\lambda^2 y}\, y^{\frac{1}{2}q-1}\, dy$$

$$= \frac{\lambda^n}{2^{n+1}\,\Pi(n-\frac{1}{2}q)} \cdot \frac{2^q}{\lambda^q}\, \Pi(\tfrac{1}{2}q-1)$$

$$= 2^{q-n-1} \lambda^{n-q} \frac{\Pi(\frac{1}{2}q-1)}{\Pi(n-\frac{1}{2}q)}. \qquad 163$$

By putting $q = n+1$, we obtain

$$\int_0^\infty J_n(\lambda r)\, dr = \frac{1}{\lambda}; \qquad 164$$

and by putting $q = n$, which is permissible so long as n is not zero, we find

$$\int_0^\infty \frac{J_n(\lambda r)}{r}\, dr = \frac{1}{n}. \qquad 165$$

Weber's results have been independently confirmed and generalised by Gegenbauer and Sonine, especially by the latter, whose elaborate memoir (*Math. Ann.* XVI.) contains a large number of very elegant and remarkable formulæ. Some of these may be verified by special artifices adapted to the particular cases considered; but to do this would convey no idea of the author's general method of procedure, which is based upon the theory of complex integration. The formulæ obtained are connected by a very close chain of deduction, so that an intelligible account of them would involve the reproduction of a great part of the memoir, and for this we have not space. We are therefore reluctantly compelled to content ourselves with referring the reader to M. Sonine's original memoir.

We will now consider a remarkable formula which is analogous to that which is generally known as Fourier's Theorem. It may be stated as follows:—

If $\phi(r)$ is a function which is finite and continuous for all values of r between the limits p and q (p and q being real and positive), and if n is any real positive integer, then

$$\int_0^\infty d\lambda \int_q^p \lambda\rho\phi(\rho) J_n(\lambda\rho) J_n(\lambda r)\, d\rho = \begin{cases}\phi(r), \text{ if } q < r < p, \\ 0, \text{ if } r > p \text{ or } < q.\end{cases} \qquad 166$$

A very instructive, although perhaps not altogether rigorous, method of discussing this integral is given in Basset's *Hydrodynamics*, vol. II. p. 15; we will begin by giving this analysis in a slightly modified form.

Taking cylindrical coordinates ρ, θ, z let us suppose that we have in the plane $z=0$ a thin material lamina bounded by the circles $\rho=q$ and $\rho=p$; suppose, moreover, that each element attracts according to the law of gravitation, and that the surface density at any point is expressed by the formula

$$\sigma = \phi(\rho)\cos n\theta.$$

Then the potential of the lamina at any point (r, θ, z) is

$$V = \int_q^p \int_\theta^{\theta+2\pi} \frac{\rho\phi(\rho)\cos n\theta' d\rho\, d\theta'}{\{z^2+r^2+\rho^2-2r\rho\cos(\theta'-\theta)\}^{\frac{1}{2}}}.$$

Put
$$\theta'-\theta=\eta,$$
$$r^2+\rho^2-2r\rho\cos\eta = R^2;$$

then

$$V = \int_q^p \int_0^{2\pi} \frac{\rho\phi(\rho)\{\cos n\theta\cos n\eta - \sin n\theta \sin n\eta\}\, d\rho\, d\eta}{(z^2+R^2)^{\frac{1}{2}}}$$
$$= \cos n\theta \int_q^p \int_0^{2\pi} \frac{\rho\phi(\rho)\cos n\eta\, d\rho\, d\eta}{(z^2+R^2)^{\frac{1}{2}}}.$$

Now when z is positive

$$\frac{1}{(z^2+R^2)^{\frac{1}{2}}} = \int_0^\infty e^{-\lambda z} J_0(\lambda R)\, d\lambda$$

(see p. 72); hence the potential on the positive side of the plane $z=0$ is

$$V_1 = \cos n\theta \int_q^p d\rho \int_0^{2\pi} d\eta \int_0^\infty e^{-\lambda z}\rho\phi(\rho)\cos n\eta\, J_0(\lambda R)\, d\lambda. \qquad 167$$

By Neumann's formula (p. 27) we have

$$J_0(\lambda R) = J_0(\lambda r) J_0(\lambda\rho) + 2\sum_1^\infty J_s(\lambda r) J_s(\lambda\rho)\cos s\eta,$$

and hence, *if we assume that V_1 remains the same if we change the order of integration,*

$$V_1 = \cos n\theta \int_0^\infty d\lambda \int_q^p d\rho \int_0^{2\pi} e^{-\lambda z} \rho\phi(\rho) \cos n\eta \, J_0(\lambda R) \, d\eta$$

$$= 2\pi \cos n\theta \int_0^\infty d\lambda \int_q^p d\rho e^{-\lambda z} \rho\phi(\rho) J_n(\lambda r) J_n(\lambda\rho).$$

By making a similar assumption we find that the potential on the negative side of the plane $z = 0$ is

$$V_2 = 2\pi \cos n\theta \int_0^\infty d\lambda \int_q^p d\rho e^{\lambda z} \rho\phi(\rho) J_n(\lambda r) J_n(\lambda\rho).$$

Assuming that the values of $\dfrac{\partial V_2}{\partial z}$ and $\dfrac{\partial V_1}{\partial z}$ can be found by differentiating under the integral sign, and that the results are valid when $z = 0$, we have

$$\left(\frac{\partial V_2}{\partial z} - \frac{\partial V_1}{\partial z}\right)_{z=0} = 4\pi \cos n\theta \int_0^\infty d\lambda \int_q^p d\rho \, \lambda\rho \, \phi(\rho) J_n(\lambda r) J_n(\lambda\rho). \quad 168$$

Now, if r lies between p and q, the value of the expression on the left hand is

$$4\pi\sigma = 4\pi\phi(r) \cos n\theta;$$

in all other cases it is zero, except possibly when $r = q$ or $r = p$. Thus, if we admit the validity of the process by which 168 has been deduced from 167, the proposition immediately follows.

With regard to the case when $r = q$ or p it may be observed that just as the equation

$$\left(\frac{\partial V_2}{\partial z} - \frac{\partial V_1}{\partial z}\right)_{z=0} = 4\pi\sigma \qquad [q < r < p]$$

is really connected with the fact that the attraction of a uniform circular disc of surface density σ is altered (algebraically) by an amount $4\pi\sigma$ as we pass from a point close to the centre on one side to a point close to the centre on the other, so we may infer that when $r = q$ or p

$$\left(\frac{\partial V_2}{\partial z} - \frac{\partial V_1}{\partial z}\right)_{z=0} = 2\pi\sigma$$

because the expression on the left is ultimately the difference in the attraction, normal to its plane, of a uniform semicircle as we pass from a point close to the middle point of its bounding

diameter on one side of it to the corresponding point on the other*. Thus we are led to the conclusion that

$$\left.\begin{array}{c}\int_0^\infty d\lambda \int_q^p \lambda\rho\,\phi(\rho)\,J_n(\lambda\rho)\,J_n(\lambda r)\,d\rho = \tfrac{1}{2}\phi(r)\\ \text{if} \qquad r = q \text{ or } p.\end{array}\right\} \qquad 166'$$

This is what the analogy of the ordinary Fourier series would lead us to expect.

The assumptions involved in the foregoing analysis are not very easy to justify, especially the first one: we will therefore give the outline of a more satisfactory demonstration, derived, in substance, from the memoirs of Hankel (*Math. Ann.* VIII. 471) and du Bois-Reymond (*Crelle* LXIX. 82).

Consider the integral

$$u = \int_0^h d\lambda \int_r^p \lambda\rho\, J_n(\lambda r)\, J_n(\lambda\rho)\, d\rho$$

where n is a positive integer, and r, p, h are positive quantities, with $p > r > 0$.

If p is increased by a small amount dp, the corresponding change in u is

$$du = \int_0^h d\lambda\, \{\lambda p\, J_n(\lambda r)\, J_n(\lambda p)\, dp\}$$

$$= p\,dp \int_0^h \lambda J_n(r\lambda)\, J_n(p\lambda)\, d\lambda$$

$$= \frac{hp\,dp}{p^2 - r^2}\{rJ_n(ph)\,J_n'(rh) - pJ_n'(ph)\,J_n(rh)\}$$

by 107.

Now suppose that h is very large, and let

$$\alpha = \frac{(2n+1)\pi}{4};$$

then we may put

$$J_n(ph) = \sqrt{\frac{2}{\pi ph}}\cos(\alpha - ph),$$

$$J_n'(ph) = \sqrt{\frac{2}{\pi ph}}\sin(\alpha - ph),$$

* This argument applies to any disc at a point on the circumference where the curvature is continuous.

and similarly for $J_n(rh)$, $J_n'(rh)$; therefore ultimately when h is very large,

$$du = \frac{hp\,dp}{p^2 - r^2} \cdot \frac{2}{\pi h\sqrt{rp}} \{r\cos(\alpha - ph)\sin(\alpha - rh)$$
$$- p\sin(\alpha - ph)\cos(\alpha - rh)\}$$
$$= \frac{p^{\frac{1}{2}}\,dp}{\pi r^{\frac{1}{2}}} \left\{\frac{\sin(p-r)h}{p-r} - \frac{\sin(2\alpha - (p+r)h)}{p+r}\right\}$$
$$= \frac{p^{\frac{1}{2}}\,dp}{\pi r^{\frac{1}{2}}} \left\{\frac{\sin(p-r)h}{p-r} + (-)^{n+1}\frac{\cos(p+r)h}{p+r}\right\}.$$

Hence the value of u when h is infinite is given by

$$\pi r^{\frac{1}{2}} U = \underset{h=\infty}{\mathrm{Lt}} \left\{\int_r^p \frac{\sin(p-r)h}{p-r} p^{\frac{1}{2}}\,dp\right.$$
$$\left. + (-)^{n+1}\int_r^p \frac{\cos(p+r)h}{p+r} p^{\frac{1}{2}}\,dp\right\}.$$

In virtue of two theorems due to Dirichlet

$$\underset{h=\infty}{\mathrm{Lt}} \int_r^p \frac{\sin(p-r)h}{p-r} p^{\frac{1}{2}}\,dp = \mathrm{Lt}\int_0^{p-r} \frac{\sin hx}{x}(x+r)^{\frac{1}{2}}\,dx$$
$$= \frac{\pi}{2} r^{\frac{1}{2}},$$

and $$\underset{h=\infty}{\mathrm{Lt}} \int_r^p \frac{\cos(p+r)h}{p+r} p^{\frac{1}{2}}\,dp = \mathrm{Lt}\int_{2r}^{p+r} \frac{\cos hx}{x}(x-r)^{\frac{1}{2}}\,dx$$
$$= 0;$$

consequently $$\pi r^{\frac{1}{2}} U = \tfrac{1}{2}\pi r^{\frac{1}{2}}$$

and therefore $U = \frac{1}{2}$. That is to say

$$\int_0^\infty d\lambda \int_r^p \lambda\rho\, J_n(\lambda r)\, J_n(\lambda\rho)\, d\rho = \tfrac{1}{2}.$$

In the same way it may be shown that if $0 < q < r$

$$\int_0^\infty d\lambda \int_q^r \lambda\rho\, J_n(\lambda r)\, J_n(\lambda\rho)\, d\rho = \tfrac{1}{2}.$$

It has been shown by Hankel that this last formula holds good even when $q = 0$.

It may be thought paradoxical that the value of the integral

$$\int_0^\infty d\lambda \int_r^p \lambda\rho\, J_n(\lambda r)\, J_n(\lambda\rho)\, d\rho,$$

which vanishes when $p=r$, should be independent of the upper limit p. The fact is that this statement is true only on the supposition that p and r are separated by a *finite* interval, however small that may be. Thus it may be verified that if the positive proper fraction ϵ be so determined that

$$\int_0^{\epsilon\pi} \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi$$

then
$$\operatorname*{Lt}_{h=\infty} \int_0^h d\lambda \int_r^{r+\frac{\epsilon\pi}{h}} \lambda\rho\, J_n(\lambda r)\, J_n(\lambda\rho)\, d\rho = \tfrac{1}{2},$$

so that the value of the whole integral is obtained by confining the variation of ρ to the *infinitesimal* range $(r, r+\epsilon\pi/h)$.

It should be observed that the expression

$$\int_0^h d\lambda \int_r^{r+\epsilon} \lambda\rho\, J_n(\lambda r)\, J_n(\lambda\rho)\, d\rho$$

when h becomes infinite and ϵ infinitesimal has no particular meaning unless a relation is assigned connecting the ways in which h and ϵ respectively become infinite and infinitesimal.

It may be worth noticing that these results are analogous to the reduction of the effective portion of a plane wave of light to that of a part of the first Huygens zone.

For convenience, let us write

$$\int_0^h \lambda\rho\, J_n(\lambda r)\, J_n(\lambda\rho)\, d\lambda = \Phi(\rho, h);$$

then it follows from what has just been proved that

$$\operatorname*{Lt}_{h=\infty} \int_q^p \Phi(\rho, h)\, d\rho = \begin{cases} 1 \text{ if } q<r<p, \\ \tfrac{1}{2} \text{ ,, } r=q \text{ or } r=p, \\ 0 \text{ ,, } r<q \text{ or } r>p. \end{cases}$$

In the first case of this proposition the inequalities $q<r<p$ must be understood to mean that $r-q$ and $p-r$ are *finite* positive quantities; and in like manner with regard to the other inequalities which occur in this connexion.

Now suppose that $f(\rho)$ is any function of ρ which throughout the interval (q, p) remains finite and continuous and is always increasing or always decreasing as ρ goes from q to p. Then by a lemma due to du Bois-Reymond

$$\int_q^p f(\rho)\,\Phi(\rho, h)\,d\rho = f(q)\int_q^p \Phi(\rho, h)\,d\rho$$
$$+ \{f(p) - f(q)\}\int_\mu^p \Phi(\rho, h)\,d\phi,$$

where μ is some quantity between q and p.

Suppose that r is outside the interval (q, p); then, when h becomes infinite, both the integrals on the right hand vanish, and therefore

$$\underset{h=\infty}{\mathrm{Lt}} \int_q^p f(\rho)\,\Phi(\rho, h)\,d\rho = 0 \quad [r > p \text{ or } r < q]. \qquad 169$$

Next suppose $r = q$; then by the same lemma

$$\int_r^p f(\rho)\,\Phi(\rho, h)\,d\rho = f(r)\int_r^p \Phi(\rho, h)\,d\rho + \{f(p) - f(r)\}\int_\mu^p \Phi(\rho, h)\,d\rho$$

and therefore

$$\underset{h=\infty}{\mathrm{Lt}} \int_r^p f(\rho)\,\Phi(\rho, h)\,d\rho = \tfrac{1}{2}f(r) + M\{f(p) - f(r)\},$$

where M is certainly finite, because it is zero if μ is ultimately separated from r by a finite interval, and it is $\frac{1}{2}$ when $\mu = r$.

Now by 169, if t is a positive quantity greater than p,

$$\underset{h=\infty}{\mathrm{Lt}} \int_p^t f(\rho)\,\Phi(\rho, h)\,d\rho = 0,$$

and hence, by adding the last two formulæ,

$$\underset{h=\infty}{\mathrm{Lt}} \int_r^t f(\rho)\,\Phi(\rho, h)\,d\rho = \tfrac{1}{2}f(r) + M\{f(p) - f(r)\}.$$

Since the expression on the left hand is independent of p, the same must be true of that on the right, and consequently M must be zero. Therefore

$$\underset{h=\infty}{\mathrm{Lt}} \int_r^p f(\rho)\,\Phi(\rho, h)\,d\rho = \tfrac{1}{2}f(r) \qquad 170$$

provided $r < p$ and that r and p are separated by a finite interval.

It may be proved in the same way that

$$\underset{h=\infty}{\mathrm{Lt}} \int_q^r f(\rho)\,\Phi(\rho, h)\,d\rho = \tfrac{1}{2}f(r) \qquad 170'$$

when r exceeds q by a finite quantity.

Hence, finally,

$$\operatorname*{Lt}_{h=\infty}\int_q^p d\rho\int_0^h \lambda\rho f(\rho)\,J_n(\lambda r)\,J_n(\lambda\rho)\,d\lambda=\begin{cases} f(r) & \text{if } q<r<p,\\ \tfrac{1}{2}f(r) & ,\ r=q \text{ or } r=p,\\ 0 & ,\ r<q \text{ or } r>p,\end{cases}\tag{171}$$

and, as in the other case, we may, if we like, change the order of integration.

It has been supposed that $f(\rho)$ continually increases or continually decreases throughout the interval (q, p); but it may be shown, as was done by Dirichlet in the analogous case of the Fourier integrals, that the formula is still valid if this restriction is removed. The function $f(\rho)$ may even present any finite number of isolated discontinuities in the interval, and it may become infinite for isolated values of ρ provided that

$$\int_q^p f(\rho)\,d\rho$$

is finite.

It is upon the formula 171 that Hankel bases his proof of the validity of the Fourier-Bessel Expansions briefly discussed in Chap. VI.

CHAPTER IX.

THE RELATION OF THE BESSEL FUNCTIONS TO SPHERICAL HARMONICS.

THERE is a remarkable connexion between the Bessel functions and spherical harmonics which appears to have been first discovered by Mehler (*Crelle* LXVIII. (1868), p. 134, and *Math. Ann.* v. (1872), pp. 135, 141). His results have since been developed by Heine, Hobson and others.

If, as usual, $P_n(x)$ denotes the zonal harmonic of the nth order, it is known that

$$P_n(\cos\theta) = \cos^n\theta\left\{1 - \frac{n(n-1)}{2^2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{2^2 \cdot 4^2}\tan^4\theta - \dots\right\}.$$

Put $\theta = x/n$, and suppose that n becomes infinite while x remains finite; then since

$$\operatorname*{Lt}_{n=\infty} \cos^n \frac{x}{n} = 1$$

and

$$\operatorname*{Lt}_{n=\infty} n \tan\frac{x}{n} = x,$$

it follows that

$$\operatorname*{Lt}_{n=\infty} P_n\left(\cos\frac{x}{n}\right) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \dots$$

$$= J_0(x).$$ 172

This result may also be obtained from the formula

$$\pi P_n(\cos\theta) = \int_0^\pi \{\cos\theta + i\sin\theta\cos\phi\}^n\, d\phi\,;$$

for the limit, when n is infinite, of

$$\left(\cos\frac{x}{n} + i\sin\frac{x}{n}\cos\phi\right)^n$$

is the same as that of

$$\left(1 + \frac{ix\cos\phi}{n}\right)^n,$$

and this is $e^{ix\cos\phi}$; so that

$$\pi \operatorname{Lt} P_n\left(\cos\frac{x}{n}\right) = \int_0^\pi e^{ix\cos\phi}\,d\phi$$

$$= \pi J_0(x),$$

and hence

$$\operatorname{Lt} P_n\left(\cos\frac{x}{n}\right) = J_0(x)$$

as before.

Another known theorem (Heine, *Kugelfunctionen* I., p. 165) is that

$$Q_n(\cosh\theta) = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\phi}{(\cosh\theta + \sinh\theta\cosh\phi)^{n+1}},$$

and if we put $\theta = x/n$, and proceed as before, we find

$$\operatorname{Lt} Q_n\left(\cosh\frac{x}{n}\right) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x\cosh\phi}\,d\phi$$

$$= K_0(x) \qquad 173$$

(see p. 67).

Changing x into ix, we obtain

$$\left.\begin{aligned} &\operatorname{Lt} P_n\left(\cosh\frac{x}{n}\right) = I_0(x), \\ &\operatorname{Lt} Q_n\left(\cos\frac{x}{n}\right) = (\log 2 - \gamma) J_0(x) - Y_0(x) \end{aligned}\right\} \qquad 174$$

(*Kugelf.* I., pp. 184, 245*).

Thus every theorem in zonal harmonics may be expected to yield a corresponding theorem in Bessel functions of zero order: it may, of course, happen in particular cases that the resulting theorem is a mere identity, or presents itself in an indeterminate form which has to be evaluated.

* There is a misprint on p. 245: in line 8 read C instead of $-C$.

Heine has shown that the Bessel functions of higher orders may be regarded as limiting cases of the associated functions which he denotes by $P_\nu^n(x)$. For the sake of a consistent notation we shall write $P_n^\nu(x)$ instead of Heine's $P_\nu^n(x)$.

Then (*Kugelf.* I. p. 207)

$$\pi P_n^s(\cos\theta)$$
$$= 2^n \frac{\Pi(n+s)\,\Pi(n-s)}{\Pi(2n)} \int_0^\pi (\cos\theta + i\sin\theta\cos\phi)^n \cos s\phi\, d\phi,$$

and therefore

$$\pi \mathop{\mathrm{Lt}}_{n=\infty} P_n^s\left(\cos\frac{x}{n}\right)$$
$$= \mathop{\mathrm{Lt}}_{n=\infty} \left\{2^n \frac{\Pi(n+s)\,\Pi(n-s)}{\Pi(2n)}\right\} \int_0^\pi e^{ix\cos\phi} \cos s\phi\, d\phi.$$

Now when t is very large we have asymptotically

$$\Pi(t) = \sqrt{2\pi}\, e^{-t}\, t^{t+\frac{1}{2}},$$

and therefore to the same degree of approximation, when n is large,

$$2^n \cdot \frac{\Pi(n-s)\,\Pi(n+s)}{\Pi(2n)} = \frac{2^n \cdot 2\pi e^{-2n}(n+s)^{n+s+\frac{1}{2}}(n-s)^{n-s+\frac{1}{2}}}{\sqrt{2\pi}\, 2^{2n+\frac{1}{2}}\, e^{-2n}\, n^{2n+\frac{1}{2}}}$$
$$= \frac{\sqrt{n\pi}}{2^n}\left(1+\frac{s}{n}\right)^{n+s+\frac{1}{2}}\left(1-\frac{s}{n}\right)^{n-s+\frac{1}{2}},$$

hence we infer that

$$\mathop{\mathrm{Lt}}_{n=\infty} \left\{\frac{2^{2n}}{\sqrt{n\pi}} \frac{\Pi(n-s)\,\Pi(n+s)}{\Pi(2n)}\right\} = \mathop{\mathrm{Lt}}_{n=\infty} \left(1+\frac{s}{n}\right)^{n+s+\frac{1}{2}}\left(1-\frac{s}{n}\right)^{n-s+\frac{1}{2}}$$
$$= e^s \cdot e^{-s} = 1.$$

Consequently

$$\mathop{\mathrm{Lt}}_{n=\infty} \left\{\frac{2^n}{\sqrt{n\pi}} P_n^s\left(\cos\frac{x}{n}\right)\right\} = \frac{1}{\pi}\int_0^\pi e^{ix\cos\phi}\cos s\phi\, d\phi$$
$$= i^s J_s(x), \qquad 175$$

which is Heine's formula (*l. c.* I., p. 232).

In the same way it may be inferred from the formula (*l. c.*, p. 223)

$$Q_n^s(\cosh\theta) = \frac{\Pi(2n+1)}{2^n\Pi(n+s)\,\Pi(n-s)} \int_0^\infty \frac{\cosh s\phi\, d\phi}{(\cosh\theta + \sinh\theta\cosh\phi)^{n+1}},$$

that

$$\underset{n=\infty}{\mathrm{Lt}}\left\{\frac{1}{2^{n+1}}\sqrt{\frac{\pi}{n}}\,Q_n^s\left(\cosh\frac{x}{n}\right)\right\}=\int_0^\infty e^{-x\cosh\phi}\cosh s\phi\,d\phi$$

$$=(-)^s K_s(x). \qquad 176$$

By changing x into ix we may obtain similar formulæ for $I_s(x)$ and

$$(\log 2-\gamma)\,J_s(x)-Y_s(x).$$

In Chap. VII. it has been supposed that the argument of the functions $I_s(x)$, $K_s(x)$ is a real quantity; in fact the functions were expressly introduced to meet the difficulties arising from the values of $J_s(x)$, $Y_s(x)$ when x is a pure imaginary.

In the *Kugelfunctionen* Heine practically defines his $K_s(x)$ by the formula

$$\left.\begin{aligned}K_s(x+0i)&=(-i)^s\int_0^\infty e^{ix\cosh\phi}\cosh s\phi\,d\phi\\&=(-)^s K_s(-x-0i),\\ \text{where } x+0i&=a(\sin\alpha+i\cos\alpha),\quad -\tfrac{1}{2}\pi<\alpha<\tfrac{1}{2}\pi;\end{aligned}\right\} \qquad 177$$

hence if, to avoid confusion, we write $\mathfrak{K}_s(x)$ for Heine's $K_s(x)$ and use the latter symbol for an extended definition of the function previously denoted by it, we may write

$$\left.\begin{aligned}K_s(x)&=(-)^s\int_0^\infty e^{-x\cosh\phi}\cosh s\phi\,d\phi\\&=i^{-s}\,\mathfrak{K}_s(ix),\\ \text{with}\quad x&=a(\cos\alpha+i\sin\alpha),\quad -\tfrac{1}{2}\pi<\alpha<\tfrac{1}{2}\pi.\end{aligned}\right\} \qquad 178$$

By putting

$$K_s(-x)=(-)^s K_s(x) \qquad 179$$

we give a meaning to $K_s(x)$ in all cases except when x is a pure imaginary; and if we define $K_s(ti)$, where t is real, by the formula

$$K_s(ti)=\tfrac{1}{2}\{K_s(ti+0)+K_s(ti-0)\},$$

we find that

$$K_s(ti)=i^s\{(\log 2-\gamma)\,J_s(t)-Y_s(t)\}. \qquad 180$$

From our present point of view, the most proper course is to take as the standard solutions of Bessel's equation, not J_n and Y_n, but J_n and $i^{-n}K_n(ix)$ as above defined. The function $i^{-n}K_n(ix)$ has in fact already appeared in Chap. VII.; if, when it is made

one-valued as explained above, we call it $G_n(x)$, then $Y_n(x)$ also becomes a one-valued function defined by

$$Y_n(x) = (\log 2 - \gamma)\, J_n(x) - G_n(x) \qquad 181$$

with a discontinuity along the axis of real quantities expressed by the formula

$$Y_n(t + 0i) - Y_n(t - 0i) = -\pi i\, J_n(t).$$

For any point on the axis of real quantities

$$Y_n(t) = (\log 2 - \gamma)\, J_n(t) - G_n(t). \qquad 182$$

The reader will observe that, in fact, $G_n(x)$ is identical with Heine's $K_n(x)$. We have employed $K_n(x)$ in a different sense in conformity with the usage now generally current in England: but it must be admitted that Heine's notation is, on the whole, the preferable one.

To Heine is also due the following excellent illustration of the methods of this chapter.

It is known that, with a proper determination of the signs of the radicals (see *Kugelf.* I. p. 49),

$$P_n(xy - \sqrt{x^2-1}\,.\,\sqrt{y^2-1}\cos\phi) = \sum_0^n (-)^s a_s P_n^s(x)\, P_n^s(y) \cos s\phi, \qquad 183$$

where the summation refers to s, and

$$a_0 = \frac{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2}{(n\,!)^2}$$

$$a_s = 2\,.\,\frac{\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2}{(n-s)\,!\,(n+s)\,!} \qquad [s > 0].$$

Now suppose that

$$x = \cos\frac{b}{n}, \quad y = \cos\frac{c}{n},$$

where b, c are finite real quantities, and that n is very large.

Then $\qquad \sqrt{x^2-1} = i\sin\frac{b}{n}, \quad \sqrt{y^2-1} = i\sin\frac{c}{n}$

and

$$\begin{aligned} xy - \sqrt{x^2-1}\,.\,\sqrt{y^2-1}\cos\phi &= \cos\frac{b}{n}\cos\frac{c}{n} + \sin\frac{b}{n}\sin\frac{c}{n}\cos\phi \\ &= 1 - \frac{b^2 - 2bc\cos\phi + c^2}{2n^2} + \ldots \\ &= \cos\frac{a}{n} \end{aligned}$$

to the second order, if

$$a = \sqrt{b^2 - 2bc\cos\phi + c^2}.$$

Making n increase indefinitely, and employing 183, 175, we have

$$J_0\sqrt{(b^2 - 2bc\cos\phi + c^2)} = \sum_0^\infty A_s J_s(b) J_s(c)\cos s\phi,$$

where
$$A_0 = \operatorname*{Lt}_{n=\infty} \frac{\{1.3.5\ldots(2n-1)\}^2}{(n!)^2}\frac{n\pi}{2^{2n}}$$

$$\tfrac{1}{2}A_s = \operatorname*{Lt}_{n=\infty} \frac{\{1.3.5\ldots(2n-1)\}^2}{(n+s)!\,(n-s)!}\frac{n\pi}{2^{2n}}.$$

Proceeding as on p. 89, we find

$$A_0 = \operatorname*{Lt}_{n=\infty} \frac{(2n!)^2\, n\pi}{2^{4n}\,(n!)^4} = \operatorname{Lt} \frac{\{\sqrt{2\pi}\, e^{-2n}(2n)^{2n+\frac{1}{2}}\}^2 n\pi}{2^{4n}\,\{\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}\}^4} = 1;$$

and similarly

$$\tfrac{1}{2}A_s = \operatorname{Lt} \frac{(2n!)^2\, n\pi}{2^{4n}\,(n!)^2\,(n+s)!\,(n-s)!} = 1;$$

thus finally we arrive at Neumann's formula

$$J_0(\sqrt{b^2 - 2bc\cos\phi + c^2}) = J_0(b)\,J_0(c) + 2\sum_1^\infty J_s(b)\,J_s(c)\cos s\phi. \qquad 184$$

In the same way, from the expansion

$$(n+\tfrac{1}{2})\,Q_n\{xy - \sqrt{(x^2-1)(y^2-1)}\cos\phi\} = \sum_0^\infty{}' P_n^s(y)\,Q_n^s(x)\cos s\phi \qquad 185$$

(*Kugelf.* I. p. 333) we derive the formula

$$G_0(\sqrt{b^2 - 2bc\cos\phi + c^2}) = G_0(b)\,J_0(c) + 2\sum_1^\infty G_s(b)\,J_s(c)\cos s\phi, \qquad 186$$

in which it is supposed that b, c are real and that $b > c$.

By combining 186, 184, and 181 we infer that

$$Y_0(\sqrt{b^2 - 2bc\cos\phi + c^2}) = Y_0(b)\,J_0(c) + 2\sum_1^\infty Y_s(b)\,J_s(c)\cos s\phi, \qquad 187$$

a result originally given by Neumann (*Bessel'sche Functionen*, p. 65).

We shall not proceed any further with the analytical part of the theory; for the extension of spherical harmonics and Bessel

functions to a p-dimensional geometry the reader may consult Heine (*Kugelf.* I. pp. 449—479) and Hobson (*Proc. L. M. S.* XXII. p. 431, and XXV. p. 49), while for the solution of ordinary differential equations by means of Bessel functions he may be referred to Lommel's treatise, and the papers by the same author in the *Mathematische Annalen*. (On Riccati's equation, in particular, see Glaisher in *Phil. Trans.* 1881, and Greenhill, *Quart. Jour. of Math.* vol. XVI.)

CHAPTER X.

VIBRATIONS OF MEMBRANES.

ONE of the simplest applications of the Bessel functions occurs in the theory of the transverse vibrations of a plane circular membrane. By the term *membrane* we shall understand a thin, perfectly flexible, material lamina, of uniform density throughout; and we shall suppose that it is maintained in a state of uniform tension by means of suitable constraints applied at one or more closed boundaries, all situated in the same plane. When the membrane is slightly displaced from its position of stable equilibrium, and then left to itself, it will execute small oscillations, the nature of which we shall proceed to consider, under certain assumptions made for the purpose of simplifying the analysis.

We shall attend only to the transverse vibrations, and assume that the tension remains unaltered during the motion; moreover if $z = 0$ represents the plane which contains the membrane in its undisturbed position, and if

$$z = \phi(x, y)$$

defines the form of the membrane at any instant, it will be supposed that $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are so small that their squares may be neglected.

Let σ be the mass of the membrane per unit of area, and let Tds be the tension across a straight line of length ds drawn anywhere upon the membrane; moreover let dS be an element of area, which for simplicity we may suppose bounded by lines of curvature. Then if r_1, r_2 are the principal radii of curvature, the applied force on the element is

$$T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) dS$$

and its line of action is along the normal to the element. For clearness, suppose that the element is concave to the positive direction of the axis of z: then the equation of motion is

$$\sigma dS \frac{\partial^2 z}{\partial t^2} = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) dS \cos\psi,$$

where ψ is the small angle which the inward-drawn normal makes with the axis of z.

Now, neglecting squares of small quantities,

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2},$$

and $$\cos\psi = 1;$$

hence the equation of motion becomes

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right), \qquad 1$$

with $$c^2 = \frac{T}{\sigma}. \qquad 2$$

It remains to find a solution of 1 sufficiently general to satisfy the initial and boundary conditions; these are that z and $\partial z/\partial t$ may have prescribed values when $t=0$, and that $z=0$, for all values of t, at points on the fixed boundaries of the membrane.

By changing from rectangular to cylindrical coordinates the equation 1 may be transformed into

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right). \qquad 3$$

Now suppose that the membrane is circular, and bounded by the circle $r=a$; then we have to find a solution of 3 so as to satisfy the initial conditions, and such that $z=0$, when $r=a$, for all values of t.

Assume $$z = u\cos pt, \qquad 4$$

where u is independent of t; then putting

$$\frac{p}{c} = \kappa, \qquad 5$$

u has to satisfy the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \kappa^2 u = 0; \qquad 6$$

and if we further assume that

$$u = v \cos n\theta, \tag{7}$$

where v is a function of r only, this will be a solution provided that

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \left(\kappa^2 - \frac{n^2}{r^2}\right)v = 0. \tag{8}$$

It will be sufficient for our present purpose to suppose that n is a positive integer; this being so, the solution of 8 is

$$v = AJ_n(\kappa r) + BY_n(\kappa r).$$

From the conditions of the problem v must be finite when $r = 0$: hence $B = 0$, and we have a solution of 3 in the form

$$\begin{aligned} z &= AJ_n(\kappa r)\cos n\theta \cos pt \\ &= AJ_n(\kappa r)\cos n\theta \cos \kappa ct. \end{aligned} \tag{9}$$

In order that the boundary condition may be satisfied, we must have

$$J_n(\kappa a) = 0, \tag{10}$$

and this is a transcendental equation to find κ. It has been proved in Chap. V. that this equation has an infinite number of real roots $\kappa_1, \kappa_2, \kappa_3$, etc.; to each of these corresponds a normal vibration of the type 9. The initial conditions which result in this particular type of vibration and no others are that when $t = 0$,

$$z = AJ_n(\kappa r)\cos n\theta,$$

$$\frac{\partial z}{\partial t} = 0.$$

By assigning to n the values 0, 1, 2, etc. and taking with each value of n the associated quantities $\kappa_1^{(n)}, \kappa_2^{(n)}, \kappa_3^{(n)}$, etc. derived from $J_n(\kappa a) = 0$, we are enabled to construct the more general solution

$$\begin{aligned} z = \Sigma\, (A_{ns} \cos n\theta \cos \kappa_s^{(n)} ct + B_{ns} \sin n\theta \cos \kappa_s^{(n)} ct \\ + C_{ns} \cos n\theta \sin \kappa_s^{(n)} ct + D_{ns} \sin n\theta \sin \kappa_s^{(n)} ct)\, J_n(\kappa_s^{(n)} r), \end{aligned} \tag{11}$$

where $A_{ns}, B_{ns}, C_{ns}, D_{ns}$ denote arbitrary constants.

If the initial configuration is defined by

$$z = f(r, \theta)$$

we must have

$$f(r, \theta) = \Sigma\, (A_{ns} \cos n\theta + B_{ns} \sin n\theta)\, J_n(\kappa_s^{(n)} r), \tag{12}$$

and whenever $f(r, \theta)$ admits of an expansion of this form the coefficients A_{ns}, B_{ns} are determined as in Chap. V. (pp. 56, 57) in the form of definite integrals. In fact, writing κ_s, for convenience, instead of $\kappa_s^{(n)}$,

$$\left.\begin{aligned} A_{ns} &= \frac{2}{\pi a^2 J_n'^2(\kappa_s a)} \int_0^{2\pi} d\theta \int_0^a f(r, \theta) \cos n\theta J_n(\kappa_s r)\, r dr, \\ B_{ns} &= \frac{2}{\pi a^2 J_n'^2(\kappa_s a)} \int_0^{2\pi} d\theta \int_0^a f(r, \theta) \sin n\theta J_n(\kappa_s r)\, r dr. \end{aligned}\right\} \qquad 13$$

Since $J_n(\kappa_s a) = 0$, it follows from the formula 19 (p. 13) that

$$J_n'(\kappa_s a) = J_{n-1}(\kappa_s a),$$

so that we may put $J_{n-1}^2(\kappa_s a)$ for $J_n'^2(\kappa_s a)$ in the expressions for A_{ns}, B_{ns}.

If the membrane starts from rest, the coefficients C_{ns}, D_{ns} are all zero. If, however, we suppose, for the sake of greater generality, that the initial motion is defined by the equation

$$\left(\frac{\partial z}{\partial t}\right)_0 = \phi(r, \theta),$$

we must have

$$\phi(r, \theta) = \Sigma \kappa_s^{(n)} c\, (C_{ns} \cos n\theta + D_{ns} \sin n\theta)\, J_n(\kappa_s^{(n)} r), \qquad 14$$

from which the coefficients C_{ns}, D_{ns} may be determined.

From the nature of the case the functions $f(r, \theta)$, $\phi(r, \theta)$ are one-valued, finite, and continuous, and are periodic in θ, the period being 2π or an aliquot part of 2π; thus $f(r, \theta)$,—and in like manner $\phi(r, \theta)$,—may be expanded in the form

$$\begin{aligned} f(r, \theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \dots \\ + b_1 \sin\theta + b_2 \sin 2\theta + \dots, \end{aligned}$$

the quantities a_s, b_s being functions of r. The possibility of expanding these functions in series of the form $\Sigma A_s J_n(\kappa_s r)$ has been already considered in Chaps. VI. and VIII.

In order to realise more clearly the character of the solution thus obtained, let us return to the normal oscillation corresponding to

$$z = J_n(\kappa_s r) \cos n\theta \cos \kappa_s ct, \qquad 15$$

κ_s being the sth root of $J_n(\kappa_s a) = 0$.

Each element of the membrane executes a simple harmonic oscillation of period

$$\frac{2\pi}{\kappa_s c} = \frac{2\pi}{\kappa_s}\sqrt{\frac{\sigma}{T}},$$

and of amplitude

$$J_n(\kappa_s r)\cos n\theta.$$

The amplitude vanishes, and the element accordingly remains at rest, if

$$J_n(\kappa_s r) = 0,$$

or if

$$\cos n\theta = 0.$$

The first equation is satisfied, not only when $r = a$, that is at the boundary, but also when

$$r = \frac{\kappa_1}{\kappa_s}a, \quad r = \frac{\kappa_2}{\kappa_s}a, \ldots \; r = \frac{\kappa_{s-1}}{\kappa_s}a;$$

consequently there exists a series of $(s-1)$ nodal circles concentric with the fixed boundary.

The second equation, $\cos n\theta = 0$, is satisfied when

$$\theta = \frac{\pi}{2n}, \quad \theta = \frac{3\pi}{2n}, \ldots \; \theta = \frac{(4n-1)\pi}{2n};$$

therefore there is a system of n nodal diameters dividing the membrane into $2n$ equal segments every one of which vibrates in precisely the same way. It should be observed, however, that at any particular instant two adjacent segments are in opposite phases.

The normal vibration considered is a possible form of oscillation not only for the complete circle but also for a membrane bounded by portions of the nodal circles and nodal diameters.

It is instructive to notice the dimensions of the quantities which occur in the equations. The dimensional formula for σ is

$$[\sigma] = [\mathbf{ML^{-2}}];$$

that for T is

$$[T] = [\mathbf{MT^{-2}}];$$

hence by 2 that of c is

$$[c] = [\mathbf{LT^{-1}}].$$

Since κ_s is found from $J_n(\kappa_s a) = 0$, $\kappa_s a$ is an abstract number, and

$$[\kappa_s] = [\mathbf{L^{-1}}].$$

Thus the period $2\pi/\kappa_s c$ comes out, as of course it should, of dimension [**T**].

If we write μ_s for $\kappa_s a$, so that μ_s is the sth root of $J_n(x)=0$, the period may be written in the form

$$\frac{2\pi a}{\mu_s}\sqrt{\frac{\sigma}{T}}=\frac{2}{\mu_s}\sqrt{\frac{\pi M}{T}}, \qquad 16$$

where M is the mass of the whole membrane. This shows very clearly how the period is increased by increasing the mass of the membrane, or diminishing the tension to which it is subjected.

As a particular case, suppose $n=0$, and let $\mu_1=2\cdot4048$, the smallest root of $J_0(x)=0$; then we have the gravest mode of vibration which is symmetrical about the centre, and its frequency is

$$\frac{\mu_1}{2\sqrt{\pi}}\sqrt{\frac{T}{M}}=\sqrt{\frac{T}{M}}\times\cdot678389.$$

Thus, for instance, if a circular membrane 10 cm. in diameter and weighing ·006 grm. per square cm. vibrates in its gravest mode with a frequency 220, corresponding to the standard A adopted by Lord Rayleigh, the tension T is determined by

$$\sqrt{\left(\frac{T}{25\pi\times\cdot006}\right)}\times\cdot6784=220,$$

whence

$$T=\left(\frac{220}{\cdot6784}\right)^2\times\cdot15\pi=49560$$

in dynes per centimetre, approximately. In gravitational units of force this is about 50 grams per centimetre, or, roughly, 3·4 lb. per foot.

In the case of an annular membrane bounded by the circles $r=a$ and $r=b$, the normal type of vibration will generally involve both Bessel and Neumann functions. Thus if we put

$$z=A\left\{\frac{J_n(\kappa r)}{J_n(\kappa a)}-\frac{Y_n(\kappa r)}{Y_n(\kappa a)}\right\}\cos n\theta\cos\kappa ct, \qquad 17$$

this will correspond to a possible mode of vibration provided that κ is determined so as to satisfy

$$J_n(\kappa a)\,Y_n(\kappa b)-J_n(\kappa b)\,Y_n(\kappa a)=0. \qquad 18$$

It may be inferred from the asymptotic values of J_n and Y_n that this equation has an infinite number of real roots; and it seems probable that the solution

$$z = \Sigma\Sigma \{A \cos n\theta + B \sin n\theta\} \left\{\frac{J_n(\kappa r)}{J_n(\kappa a)} - \frac{Y_n(\kappa r)}{Y_n(\kappa a)}\right\} \cos \kappa ct \qquad 19$$

is sufficiently general to meet the case when the membrane starts from rest in the configuration defined by

$$z = f(r, \theta).$$

Assuming that this is so, the coefficients A, B can be expressed in the form of definite integrals by a method precisely similar to that explained in Chap. VI. Thus if we write

$$u = \frac{J_n(\kappa r)}{J_n(\kappa a)} - \frac{Y_n(\kappa r)}{Y_n(\kappa a)},$$

it will be found that

$$\left.\begin{aligned} \int_0^{2\pi} d\theta \int_a^b f(r, \theta)\, ur \cos n\theta\, dr = LA, \\ \int_0^{2\pi} d\theta \int_a^b f(r, \theta)\, ur \sin n\theta\, dr = LB, \end{aligned}\right\} \qquad 20$$

where

$$L = \pi \left[\frac{r^2}{2} \left\{\frac{J_n'(\kappa r)}{J_n(\kappa a)} - \frac{Y_n'(\kappa r)}{Y_n(\kappa a)}\right\}^2\right]_a^b. \qquad 21$$

This value of L may perhaps be reducible to a simpler form in virtue of the condition

$$J_n(\kappa a)\, Y_n(\kappa b) - J_n(\kappa b)\, Y_n(\kappa a) = 0.$$

For a more detailed treatment of the subject of this chapter the reader is referred to Riemann's *Partielle Differentialgleichungen* and Lord Rayleigh's *Theory of Sound.*

CHAPTER XI.

HYDRODYNAMICS.

IN Chapter VI. it has been shown that the expression

$$\phi = \Sigma (A \cos n\theta + B \sin n\theta) e^{-\lambda z} J_n(\lambda r)$$

satisfies Laplace's equation

$$\nabla^2\phi = 0,$$

and some physical applications of this result have been already considered. In the theory of fluid motion ϕ may be interpreted as a velocity-potential defining a form of steady irrotational motion of an incompressible fluid, and is a proper form to assume when we have to deal with cylindrical boundaries.

We shall not stay to discuss any of the special problems thus suggested, but proceed to consider some in which the method of procedure is less obvious.

Let there be a mass of incompressible fluid of unit density moving in such a way that the path of each element lies in a plane containing the axis of z, and that the molecular rotation is equal to ω, the axis of rotation for any element being perpendicular to the plane which contains its path.

Then, taking cylindrical coordinates r, θ, z as usual, and denoting by u, v the component velocities along r and parallel to the axis of z respectively,

$$\frac{\partial}{\partial r}(ur) + \frac{\partial}{\partial z}(vr) = 0, \qquad 1$$

and

$$\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z} = 2\omega. \qquad 2$$

Equation 1 shows that we may put

$$ur = -\frac{\partial\psi}{\partial z}, \quad vr = \frac{\partial\psi}{\partial r}, \qquad 3$$

where ψ is Stokes's current function; thus equation 2 becomes

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) - 2\omega = 0. \qquad 4$$

When the motion is steady, ψ is a function of r and z; and if we put

$$q^2 = u^2 + v^2,$$

so that q is the resultant velocity, the dynamical equations may be written in the form

$$\left.\begin{aligned} \frac{\partial p}{\partial r} + \frac{\partial}{\partial r}(\tfrac{1}{2}q^2) - 2\frac{\omega}{r}\frac{\partial\psi}{\partial r} = 0, \\ \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}(\tfrac{1}{2}q^2) - 2\frac{\omega}{r}\frac{\partial\psi}{\partial z} = 0, \end{aligned}\right\} \qquad 5$$

whence it follows that ω/r must be expressible as a function of ψ. The simplest hypothesis is

$$\omega = \zeta r, \qquad 6$$

where ζ is a constant; on this assumption, 4 becomes

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) - 2\zeta r = 0. \qquad 7$$

Now the ordinary differential equation

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{dr}\right) - 2\zeta r = 0$$

is satisfied by

$$\chi = \tfrac{1}{4}\zeta r^4 + Ar^2 + B,$$

where A and B are arbitrary constants; and if we assume

$$\psi = \chi + \rho r \cos nz,$$

where ρ is a function of r only, we find from 7 that

$$\frac{d^2\rho}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} - \left(n^2 + \frac{1}{r^2}\right)\rho = 0,$$

the solution of which is

$$\rho = C_n I_1(nr) + D_n K_1(nr).$$

Finally, then,

$$\psi = \tfrac{1}{4}\zeta r^4 + Ar^2 + B + r\sum_n \{C_n I_1(nr) + D_n K_1(nr)\} \cos nz, \qquad 8$$

where the values of n and of the other constants have to be determined so as to meet the requirements of the boundary conditions.

Suppose, for instance, that the fluid fills the finite space inclosed by the cylinders $r=a$, $r=b$ and the planes $z=\pm h$. Then the boundary conditions are

$$\frac{\partial\psi}{\partial z}=0$$

when $r=a$ or b, for all values of z; and

$$\frac{\partial\psi}{\partial r}=0$$

when $z=\pm h$, for all values of r such that

$$a \geqq r \geqq b.$$

One way of satisfying these conditions is to make ψ constant and equal to zero at every point on the boundary. Now if we put

$$\psi=\tfrac{1}{4}\zeta(r^2-a^2)(r^2-b^2)-\zeta r\sum_n C_n\left\{\frac{I_1(nr)}{I_1(na)}-\frac{K_1(nr)}{K_1(na)}\right\}\frac{\cos nz}{\cos nh} \qquad 9$$

this is of the right form, and vanishes for $r=a$. It vanishes when $r=b$, provided the values of n are chosen so as to satisfy

$$I_1(na)K_1(nb)-I_1(nb)K_1(na)=0; \qquad 10$$

and, finally, it vanishes when $z=\pm h$ if the coefficients C_n are determined so that

$$\sum_n C_n\left\{\frac{I_1(nr)}{I_1(na)}-\frac{K_1(nr)}{K_1(na)}\right\}=\tfrac{1}{4}(r^2-a^2)(r^2-b^2)/r \qquad 11$$

for all values of r such that

$$a \geqq r \geqq b.$$

Assuming the possibility of this expansion, the coefficients are found in the usual way by integration.

The stream-lines are defined by

$$\psi=\text{const.},\quad \theta=\text{const.},$$

so that the outermost particles of fluid remain, throughout the motion, in contact with the containing vessel.

(The above solution was given in the Mathematical Tripos, Jan. 1884.)

Some very interesting results have been obtained by Lord Kelvin (*Phil. Mag.* (5) x. (1880), p. 155) in connexion with the oscillations of a cylindrical vortex about a state of steady motion. Adopting the fluxional notation to denote complete differentiation with respect to the time, the dynamical equations of motion, and the equation of continuity, are in cylindrical coordinates

$$\left.\begin{aligned} -\frac{\partial p}{\partial r} &= \frac{\partial \dot{r}}{\partial t} - r\dot{\theta}^2 + \dot{r}\frac{\partial \dot{r}}{\partial r} + \dot{\theta}\frac{\partial \dot{r}}{\partial \theta} + \dot{z}\frac{\partial \dot{r}}{\partial z}, \\ -\frac{\partial p}{r\partial \theta} &= r\frac{\partial \dot{\theta}}{\partial t} + \dot{r}\dot{\theta} + \dot{r}\frac{\partial (r\dot{\theta})}{\partial r} + \dot{\theta}\frac{\partial (r\dot{\theta})}{\partial \theta} + \dot{z}\frac{\partial (r\dot{\theta})}{\partial z}, \\ -\frac{\partial p}{\partial z} &= \frac{\partial \dot{z}}{\partial t} + \dot{r}\frac{\partial \dot{z}}{\partial r} + \dot{\theta}\frac{\partial \dot{z}}{\partial \theta} + \dot{z}\frac{\partial \dot{z}}{\partial z}, \end{aligned}\right\} \qquad 12$$

$$\frac{\partial \dot{r}}{\partial r} + \frac{\dot{r}}{r} + \frac{\partial (r\dot{\theta})}{r\partial \theta} + \frac{\partial \dot{z}}{\partial z} = 0. \qquad 13$$

It is to be understood that r, θ, z are treated as functions of t, while $\dot{r}$, $\dot{\theta}$, $\dot{z}$ are supposed expressed as explicit functions of r, θ, z, t; and the density of the liquid is taken to be unity.

We obtain a possible state of steady motion by supposing that

$$\dot{r} = 0, \quad \dot{z} = 0, \quad \dot{\theta} = \omega \text{ (a constant)},$$

this makes the resultant velocity

$$U = \omega r, \qquad 14$$

while the pressure is

$$\Pi = \int \omega^2 r dr = \Pi_0 + \tfrac{1}{2}\omega^2 r^2, \qquad 15$$

Π_0 being a constant depending on the boundary conditions.

Now assume as a solution of 12 and 13

$$\left.\begin{aligned} \dot{r} &= \rho \cos mz \sin (nt - s\theta), & r\dot{\theta} &= U + u \cos mz \cos (nt - s\theta), \\ \dot{z} &= w \sin mz \sin (nt - s\theta), & p &= \Pi + \varpi \cos mz \cos (nt - s\theta), \end{aligned}\right\} \qquad 16$$

where s is a real integer, m, n are constants, and ρ, u, w, ϖ are functions of r which are small in comparison with U. Then substituting in 12 and 13 and neglecting squares and products of small quantities, we obtain the approximate equations

$$\left.\begin{aligned} -\frac{d\varpi}{dr} &= (n - s\omega)\rho - 2\omega u, \\ -\frac{s\varpi}{r} &= -(n - s\omega)u + 2\omega\rho, \\ m\varpi &= (n - s\omega)w, \end{aligned}\right\} \qquad 17$$

$$\frac{d\rho}{dr} + \frac{\rho}{r} + \frac{su}{r} + mw = 0. \qquad 18$$

From equations 17 we obtain

$$\left.\begin{aligned} \rho &= \frac{(n-s\omega)\left\{(n-s\omega)\dfrac{dw}{dr}-\dfrac{2s\omega}{r}w\right\}}{m\left\{4\omega^2-(n-s\omega)^2\right\}} \\ u &= \frac{(n-s\omega)\left\{2\omega\dfrac{dw}{dr}-\dfrac{s(n-s\omega)}{r}w\right\}}{m\left\{4\omega^2-(n-s\omega)^2\right\}} \end{aligned}\right\}, \qquad 19$$

and on substituting these expressions in 18 we find, after a little reduction,

$$\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}+\left\{\frac{m^2\{4\omega^2-(n-s\omega)^2\}}{(n-s\omega)^2}-\frac{s^2}{r^2}\right\}w=0. \qquad 20$$

If the quantity

$$\frac{m^2\{4\omega^2-(n-s\omega)^2\}}{(n-s\omega)^2}$$

is positive, let it be called κ^2; if it is negative, let it be denoted by $-\lambda^2$. Then in the first case

$$w=AJ_s(\kappa r)+BY_s(\kappa r), \qquad 21$$

and in the second case

$$w=CI_s(\lambda r)+DK_s(\lambda r). \qquad 22$$

The constants must be determined by appropriate initial or boundary conditions. For instance, suppose the fluid to occupy, during the steady motion, the whole interior of the cylinder $r=a$. Then in order that, in the disturbed motion, w may be everywhere small it is necessary that $B=0$ in 20 and $D=0$ in 21.

To fix the ideas, suppose m, n, s, ω *assigned*, and that

$$4\omega^2>(n-s\omega)^2;$$

then by 19 and 21

$$\rho=\frac{A(n-s\omega)\left\{(n-s\omega)\kappa J_s'(\kappa r)-\dfrac{2s\omega}{r}J_s(\kappa r)\right\}}{m\left\{4\omega^2-(n-s\omega)^2\right\}}. \qquad 23$$

By 16 the corresponding radial velocity is

$$\dot{r}=\rho\cos mz\sin(nt-s\theta)$$

and if ρ_0 is the value of ρ when $r=a$, the initial velocity along the radius, for $r=a$, is

$$-\rho_0\cos mz\sin s\theta.$$

Now ρ_0 may have any (small) constant value; supposing that this is prescribed, the constant A is determined, its value being, by 23,

$$A = \frac{\{4\omega^2 - (n - s\omega)^2\}\, m\rho_0}{(n - s\omega)\left\{(n - s\omega)\,\kappa J_s'(\kappa a) - \dfrac{2s\omega}{a} J_s(\kappa a)\right\}}$$

$$= \frac{\kappa^2\rho_0/m}{\kappa J_s'(\kappa a) - \dfrac{2s\omega}{(n - s\omega)\,a} J_s(\kappa a)}. \qquad 24$$

Of course, the other initial component velocities and the initial pressure must be adjusted so as to be consistent with the equations 16—19.

There is no difficulty in realising the general nature of the disturbance represented by the equations 16; it evidently travels round the axis of the cylinder with constant angular velocity n/s.

When ω is given, we can obtain a very general solution by compounding the different disturbances of the type considered which arise when we take different values of m, n, s; according to Lord Kelvin it is possible to construct in this way the solution for "any arbitrary distribution of the generative disturbance over the cylindric surface, and for any arbitrary periodic function of the time."

The general solution involves both the J and the I functions.

Another case of steady motion is that of a hollow irrotational vortex in a fixed cylindrical tube. This is obtained by putting

$$\dot{r} = 0, \quad \dot{z} = 0, \quad r^2\dot{\theta} = c,$$

where c is a constant; the velocity-potential is $c\theta$, and the velocity at any point is

$$U = \frac{c}{r}. \qquad 25$$

If a is the radius of the free surface, the pressure for the undisturbed motion is

$$\Pi = \Pi_0 + \frac{1}{2}c^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right). \qquad 26$$

Putting these values of U and Π in equations 16 and proceeding as before, we find for the approximate equations corresponding to 17 and 18

$$\left.\begin{aligned} -\frac{d\varpi}{dr} &= \left(n - \frac{cs}{r^2}\right)\rho - \frac{2cu}{r^2}, \\ -\frac{s\varpi}{r} &= -\left(n - \frac{cs}{r^2}\right)u, \\ m\varpi &= \left(n - \frac{cs}{r^2}\right)w, \end{aligned}\right\} \qquad 27$$

$$\frac{d\rho}{dr} + \frac{\rho}{r} + \frac{su}{r} + mw = 0. \qquad 28$$

Hence
$$\left.\begin{aligned} u &= \frac{sw}{mr}, \quad \varpi = \frac{1}{m}\left(n - \frac{cs}{r^2}\right)w, \\ \rho &= -\frac{1}{m}\frac{dw}{dr}; \end{aligned}\right\} \qquad 29$$

and therefore the differential equation satisfied by w is

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - \left(m^2 + \frac{s^2}{r^2}\right)w = 0.$$

Consequently $\quad w = AI_s(mr) + BK_s(mr), \qquad$ 30

where A, B are arbitrary constants.

If the fixed boundary is defined by $r = b$, we must have $\dot{r} = 0$ when $r = b$; that is, by 16 and 29

$$\frac{dw}{dr} = 0 \text{ when } r = b.$$

Thus
$$w = A\left\{\frac{I_s(mr)}{I'_s(mb)} - \frac{K_s(mr)}{K'_s(mb)}\right\}. \qquad 31$$

We have still to express the condition that $p = \Pi_0$ at every point on the free surface for the disturbed motion. To do this we must find a first approximation to the form of the free surface. In the steady motion, the coordinates r, z of a particle of fluid remain invariable and

$$\dot{\theta} = \frac{c}{r^2}, \quad \text{whence } \theta = \frac{ct}{r^2}.$$

In the disturbed motion r does not differ much from its mean value r_0, and if we take the equation

$$\dot{r} = \rho \cos mz \sin(nt - s\theta)$$

we obtain a first approximation by putting $\theta = \frac{ct}{r_0^2}$, giving ρ its mean value ρ_0 and neglecting the variation of z: thus

$$\dot{r} = \rho_0 \cos mz \sin\left(n - \frac{cs}{r_0^2}\right)t$$

and therefore

$$r = r_0 - \frac{\rho_0}{n - \frac{cs}{r_0^2}} \cos mz \cos(nt - s\theta). \qquad 32$$

Putting $r_0 = a$, and writing ρ_a for the corresponding value of ρ_0, the approximate equation of the free surface is

$$r = a - \frac{\rho_a}{n - \frac{sc}{a^2}} \cos mz \cos(nt - s\theta). \qquad 33$$

Now by 16 and 26

$$p = \Pi_0 + \frac{1}{2}c^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + \varpi \cos mz \cos(nt - s\theta)$$

and the condition $p = \Pi_0$ gives, with the help of 33,

$$0 = \frac{1}{2}c^2 \left\{\frac{1}{a^2} - \frac{1}{\left[a - \frac{\rho_a}{n - \frac{sc}{a^2}} \cos mz \cos(nt - s\theta)\right]^2}\right\}$$
$$+ \varpi \cos mz \cos(nt - s\theta);$$

that is, neglecting the squares of small quantities,

$$\varpi - \frac{c^2}{a^3}\frac{\rho_a}{n - \frac{sc}{a^2}} = 0. \qquad 34$$

Also by 29

$$\varpi = \frac{1}{m}\left(n - \frac{sc}{a^2}\right) w_a,$$

$$\rho_a = -\frac{1}{m}\left(\frac{dw}{dr}\right)_{r=a};$$

thus, with the value of w given in 31, the condition 34 becomes

$$\left(n - \frac{sc}{a^2}\right)^2 \left\{\frac{I_s(ma)}{I_s'(mb)} - \frac{K_s(ma)}{K_s'(mb)}\right\}$$
$$+ \frac{mc^2}{a^3}\left\{\frac{I_s'(ma)}{I_s'(mb)} - \frac{K_s'(ma)}{K_s'(mb)}\right\} = 0. \qquad 35$$

This may be regarded as an equation to find n when the other quantities are given. If we write

$$\frac{c}{a^2} = \omega \qquad 36$$

(the angular velocity at the free surface in the steady motion), and

$$N = -ma\left\{\frac{I'_s(ma)}{I'_s(mb)} - \frac{K'_s(ma)}{K'_s(mb)}\right\} \div \left\{\frac{I_s(ma)}{I'_s(mb)} - \frac{K_s(ma)}{K'_s(mb)}\right\}, \qquad 37$$

the roots of the equation 35 are given by

$$n = \omega\,(s \pm \sqrt{N}). \qquad 38$$

N is an abstract number, which is positive whenever a, b, m are real and $b > a$. Thus the steady motion is stable in relation to disturbances of the type here considered. This might have been anticipated, from other considerations.

The interpretation of 38 is that corresponding to each set of values m, s there are two oscillations of the type 16, travelling with angular velocities

$$\omega\left(1 + \frac{\sqrt{N}}{s}\right) \text{ and } \omega\left(1 - \frac{\sqrt{N}}{s}\right)$$

respectively about the axis of the vortex.

A special case worth noticing is when $b = \infty$. In this case we must put

$$w = AK_s(mr)$$

and 37 reduces to

$$N = -ma\frac{K'_s(ma)}{K_s(ma)}.$$

The third case considered by Lord Kelvin is that of a cylindrical core rotating like a solid body and surrounded by liquid which extends to infinity and moves irrotationally, with no slip at the interface between it and the core. Thus if a is the radius of the core, we have

$$\left.\begin{array}{ll} U = \omega r & \text{when } r < a, \\ U = \dfrac{\omega a^2}{r} & \text{when } r > a, \end{array}\right\} \qquad 39$$

for the undisturbed motion.

For the disturbed motion we start as before with equations 16, and by precisely the same analysis we find

$$\left.\begin{aligned} w &= AJ_s(\kappa r) \text{ when } r < a, \\ w &= BK_s(mr) \text{ when } r > a, \\ \text{with} \quad \kappa^2 &= \frac{m^2\{4\omega^2 - (n - s\omega)^2\}}{(n - s\omega)^2}. \end{aligned}\right\} \qquad 40$$

At the interface ρ, w and ϖ must have the same value on both sides. Now by 17 and 27 it follows that the values of ϖ are the same when those of w agree; hence the two conditions to be satisfied are, by 23 and 29,

$$AJ_s(\kappa a) = BK_s(ma), \qquad 41$$

and

$$\frac{A(n - s\omega)\left\{(n - s\omega)\,\kappa J_s'(\kappa a) - \dfrac{2s\omega}{a} J_s(\kappa a)\right\}}{m\{4\omega^2 - (n - s\omega)^2\}} = -BK_s'(ma). \qquad 42$$

Eliminating A/B, we obtain, on reduction,

$$\frac{\kappa a J_s'(\kappa a)}{J_s(\kappa a)} + \frac{\kappa^2 a K_s'(ma)}{mK_s(ma)} - \frac{2s\omega}{n - s\omega} = 0,$$

or, which is the same thing,

$$\frac{m\kappa a J_s'(\kappa a)}{J_s(\kappa a)} + \frac{\kappa^2 a K_s'(ma)}{K_s(ma)} - s\sqrt{\kappa^2 + m^2} = 0, \qquad 43$$

a transcendental equation to find κ when the other constants are given. When κ is known, n is given by

$$n = \omega\left(s \pm \frac{2m}{\sqrt{\kappa^2 + m^2}}\right).$$

For a proof that the equation 43 has an infinite number of real roots, and for a more complete discussion of the three problems in question, the reader is referred to the original paper above cited.

Since the expression for $\dot{z}$ involves the factor $\sin mz$, we may, if we like, suppose that the planes $z = 0$ and $z = \pi/m$ are fixed boundaries of the fluid.

We will now consider the irrotational wave-motion of homogeneous liquid contained in a cylindrical tank of radius a and depth h. The upper surface is supposed free, and in the plane $z = 0$ when undisturbed.

The velocity-potential ϕ must satisfy the equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{44}$$

and also the boundary conditions

$$\left.\begin{aligned}\frac{\partial\phi}{\partial z}&=0 \quad \text{when} \quad z=-h,\\ \frac{\partial\phi}{\partial r}&=0 \quad \text{when} \quad r=a.\end{aligned}\right\} \tag{45}$$

These conditions are all fulfilled if we assume

$$\phi = AJ_n(\kappa r)\sin n\theta \cosh\kappa(z+h)\cos mt, \tag{46}$$

provided that κ is chosen so that

$$J_n'(\kappa a)=0. \tag{47}$$

If gravity is the only force acting, we have, as the condition for a free surface,

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}=0 \tag{48}$$

when $z=0$, neglecting small quantities of the second order; therefore

$$-m^2\cosh\kappa h+g\kappa\sinh\kappa h=0,$$

or

$$m^2=g\kappa\tanh\kappa h. \tag{49}$$

The equations 46, 47, 49 give a form of ϕ corresponding to a normal type of oscillation; when the liquid occupies the whole interior of the tank, n must be a whole number in order that ϕ may be one-valued. The equation 47 has an infinite number of roots $\kappa_1^{(n)}$, $\kappa_2^{(n)}$, etc., so that for each value of n we may write, more generally,

$$\phi=\sum_s A_s J_n(\kappa_s^{(n)}r)\cosh\kappa_s^{(n)}(z+h)\cos m_s^{(n)}t\sin n\theta, \tag{50}$$

and by compounding the solutions which arise from different integral values of n we obtain an expression for ϕ which contains a doubly infinite number of terms. Moreover instead of the single trigonometrical factor

$$A\cos mt\sin n\theta$$

in the typical term we may put

$$(A\cos mt+B\sin mt)\sin n\theta+(C\cos mt+D\sin mt)\cos n\theta,$$

where A, B, C, D are arbitrary constants.

As a simple illustration, let us take $n = 0$, and put

$$\phi = \Sigma A J_0(\kappa r) \cosh \kappa (z + h) \sin mt;$$

then if, as usual, we write η for the elevation of the free surface at any moment above the mean level,

$$\dot{\eta} = \left(\frac{\partial \phi}{\partial z}\right)_{z=0} = \Sigma \kappa A J_0(\kappa r) \sinh \kappa h \sin mt, \qquad 51$$

and since this vanishes when $t = 0$, the liquid must be supposed to start from rest. Integrating 51 with regard to t, we have a possible initial form of the free surface defined by

$$\eta = -\Sigma \frac{\kappa}{m} A \sinh \kappa h J_0(\kappa r), \qquad 52$$

the summation referring to the roots of

$$J_0'(\kappa a) = 0.$$

By the methods of Chap. VI. the solution may be adapted to suit a prescribed form of initial free surface defined by the equation

$$\eta = f(r).$$

It will be observed that in 49 κh is an abstract number; and if, in the special case last considered, we put $\kappa a = \lambda$, so that λ is a root of $J_0'(\lambda) = 0$, the period of the corresponding oscillation is

$$\frac{2\pi}{m} = 2\pi \sqrt{\frac{a}{\lambda g} \coth \frac{\lambda h}{a}}.$$

A specially interesting case occurs when a rigid vertical diaphragm, whose thickness may be neglected, extends from the axis of the tank to its circumference. If the position of the diaphragm is defined by $\theta = 0$, we must have, in addition to the other conditions,

$$\frac{\partial \phi}{\partial \theta} = 0 \text{ when } \theta = 0.$$

This excludes some, but not all, of the normal oscillations which are possible in the absence of the barrier; but besides those which can be retained, we have a new set which are obtained by supposing

$$n = k + \tfrac{1}{2},$$

where k is any integer. Thus in the simplest case, when $k = 0$, we may put

$$\phi = A J_{\frac{1}{2}}(\kappa r) \cos \frac{\theta}{2} \cosh \kappa (z + h) \cos mt, \qquad 53$$

or, which is the same thing,

$$\phi = Ar^{-\frac{1}{2}} \sin(\kappa r) \cos\frac{\theta}{2} \cosh\kappa(z+h)\cos mt,$$

with the conditions

$$\left.\begin{aligned} \tan\kappa a - \kappa a = 0,\\ m^2 = g\kappa\tanh\kappa h \end{aligned}\right\}. \qquad 54$$

The equation

$$\tan x - x = 0$$

has an infinite number of real roots, and to each of these corresponds an oscillation of the type represented by 53.

More generally, if we put

$$n = \frac{(2k+1)\pi}{2\alpha} \quad [\alpha < \tfrac{1}{2}\pi]$$

where k is any integer, the function

$$\phi = (A\cos mt + B\sin mt) J_n(\kappa r)\cosh\kappa(z+h)\sin n\theta,$$

with the conditions 47 and 49 as before, defines a normal type of oscillation in a tank of depth h bounded by the cylinder $r = a$ and the planes $\theta = \pm\alpha$.

Similar considerations apply to the vibrations of a circular membrane with one radius fixed, and of a membrane in the shape of a sector of a circle (see Rayleigh's *Theory of Sound*, I. p. 277).

Another instructive problem, due to Lord Kelvin (*Phil. Mag.* (5) X. (1880), p. 109), may be stated as follows.

A circular basin, containing heavy homogeneous liquid, rotates with uniform angular velocity ω about the vertical through its centre; it is required to investigate the oscillations of the liquid on the assumptions that the motion of each particle is infinitely nearly horizontal, and only deviates slightly from what it would be if the liquid and basin together rotated like a rigid body; and further that the velocity is always equal for particles in the same vertical.

The legitimacy of these assumptions is secured if we suppose that, if a is the radius of the basin, $\omega^2 a$ is small in comparison with g and that the greatest depth of the liquid is small in comparison with a. We shall suppose, for simplicity, that the mean depth is constant, and equal to h.

Let the motion be referred to horizontal rectangular axes which meet on the axis of rotation, and are rigidly connected with the basin. Then if u, v are the component velocities, parallel to these axes, of a particle whose coordinates are x, y, the approximate equations of motion are

$$\left.\begin{aligned}\frac{\partial u}{\partial t} - 2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t} + 2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\}. \qquad 55$$

If $h + z$ is the depth of the liquid in the vertical through the point considered, the equation of continuity is

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial t} = 0; \qquad 56$$

while the condition for a free surface leads to the equations

$$\left.\begin{aligned}\frac{\partial p}{\partial x} &= g\rho\frac{\partial z}{\partial x},\\ \frac{\partial p}{\partial y} &= g\rho\frac{\partial z}{\partial y}\end{aligned}\right\}. \qquad 57$$

If we eliminate p from 55 by means of 57 and change to polar coordinates, we obtain

$$\left.\begin{aligned}\frac{\partial u}{\partial t} - 2\omega v + g\frac{\partial z}{\partial r} &= 0,\\ \frac{\partial v}{\partial t} + 2\omega u + g\frac{\partial z}{r\partial\theta} &= 0\end{aligned}\right\}, \qquad 58$$

where u, v now denote the component velocities along the radius vector and perpendicular to it.

The equation of continuity, in the new notation, is

$$h\left(\frac{\partial u}{\partial r} + \frac{\partial v}{r\partial\theta} + \frac{u}{r}\right) + \frac{\partial z}{\partial t} = 0. \qquad 59$$

From the equations 58 we obtain

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial t^2} + 4\omega^2\right)u &= -g\frac{\partial^2 z}{\partial r\partial t} - 2\omega g\frac{\partial z}{r\partial\theta},\\ \left(\frac{\partial^2}{\partial t^2} + 4\omega^2\right)v &= 2\omega g\frac{\partial z}{\partial r} - g\frac{\partial^2 z}{r\partial\theta\partial t}\end{aligned}\right\}; \qquad 60$$

hence by operating on 59 with $\left(\frac{\partial^2}{\partial t^2} + 4\omega^2\right)$ and eliminating u, v we

obtain a differential equation in z, which, after reduction, is found to be

$$\left(\frac{\partial^2}{\partial t^2} + 4\omega^2\right)\frac{\partial z}{\partial t} = gh\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\frac{\partial z}{\partial t}. \qquad 61$$

Let us assume

$$z = \zeta \cos(m\theta - nt),$$

where m, n are constants, and ζ is a function of r only: then on substitution in 61 we find

$$\frac{d^2\zeta}{dr^2} + \frac{1}{r}\frac{d\zeta}{dr} + \left(\kappa^2 - \frac{m^2}{r^2}\right)\zeta = 0, \qquad 62$$

where

$$\kappa^2 = \frac{n^2 - 4\omega^2}{gh}. \qquad 63$$

The work now proceeds as in other similar cases already considered. Thus for instance in the simplest case, that of an open circular pond with a vertical bank, we take m to be a real integer, and put

$$z = J_m(\kappa r)\cos(m\theta - nt). \qquad 64$$

The boundary condition

$$u = 0 \text{ when } r = a$$

gives, for the determination of n, the equation

$$2m\omega J_m(\kappa a) - n\kappa a J'_m(\kappa a) = 0. \qquad 65$$

If ω^2 is small in comparison with gh, we have approximately

$$\kappa^2 = \frac{n^2}{gh}$$

and 65 becomes

$$2m\omega J_m\left(\frac{na}{\sqrt{gh}}\right) - \frac{n^2 a}{\sqrt{gh}} J'_m\left(\frac{na}{\sqrt{gh}}\right) = 0.$$

In the general case it will be found that the equations 58 and 60 are satisfied by putting

$$u = U\sin(m\theta - nt), \quad v = V\cos(m\theta - nt), \qquad 66$$

with

$$\left.\begin{aligned} U &= \frac{g}{n^2 - 4\omega^2}\left(n\frac{d\zeta}{dr} - \frac{2m\omega\zeta}{r}\right), \\ V &= \frac{g}{n^2 - 4\omega^2}\left(-2\omega\frac{d\zeta}{dr} + \frac{mn\zeta}{r}\right) \end{aligned}\right\}. \qquad 67$$

By assuming for the solution of 62

$$\zeta = AJ_m(\kappa r) + BY_m(\kappa r)$$

we obtain a value for z which may be adapted to the case of a circular pond with a circular island in the middle.

It should be remarked that the problem was suggested to Lord Kelvin by Laplace's dynamical theory of the tides: the solution is applicable to waves in a shallow lake or inland sea, if we put $\omega = \gamma \sin \lambda$, γ being the earth's angular velocity, and λ the latitude of the lake or sea, which is supposed to be of comparatively small dimensions.

We will conclude the chapter with a brief account of the application of Bessel functions to the two-dimensional motion of a viscous liquid. It may be shown, as in Basset's *Hydrodynamics*, II. p. 244, that if we suppose the liquid to be of unit density, and that no forces act, the equations of motion are

$$\left.\begin{aligned} \dot{u} - \frac{v^2}{r} &= -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right), \\ \dot{v} + \frac{uv}{r} &= -\frac{\partial p}{r\partial \theta} + \mu\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right) \end{aligned}\right\}. \qquad 68$$

If ψ is the current function,

$$u = \frac{\partial \psi}{r\partial \theta}, \quad v = -\frac{\partial \psi}{\partial r};$$

and if we put

$$\mu \nabla^2 \psi - \frac{\partial \psi}{\partial t} = \chi, \qquad 69$$

the equations of motion may be written in the form

$$\left.\begin{aligned} \dot{u} - \frac{v^2}{r} &= -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial \chi}{\partial \theta} + \frac{\partial u}{\partial t}, \\ \dot{v} + \frac{uv}{r} &= -\frac{\partial p}{r\partial \theta} - \frac{\partial \chi}{\partial r} + \frac{\partial v}{\partial t} \end{aligned}\right\}. \qquad 70$$

If the squares and products of the velocities are neglected,

$$\dot{u} = \frac{\partial u}{\partial t}, \quad \dot{v} = \frac{\partial v}{\partial t},$$

and the equations become

$$\left.\begin{aligned} \frac{\partial p}{\partial r} - \frac{1}{r}\frac{\partial \chi}{\partial \theta} &= 0, \\ \frac{\partial p}{\partial \theta} + r\frac{\partial \chi}{\partial r} &= 0 \end{aligned}\right\}. \qquad 71$$

Hence, eliminating p,

$$\frac{\partial}{\partial r}\left(r\frac{\partial\chi}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2}=0,$$

or, which is the same thing,

$$\nabla^2\chi=0. \qquad 72$$

A comparatively simple solution may be constructed by supposing that

$$\chi=0,$$

and $$\psi=\Psi e^{mti},$$

where Ψ is a function of r only. This leads to

$$\frac{d^2\Psi}{dr^2}+\frac{1}{r}\frac{d\Psi}{dr}-\frac{mi}{\mu}\Psi=0,$$

and if we write

$$\frac{m}{2\mu}=\kappa^2,$$

one value of Ψ is

$$\Psi=AJ_0\{(1-i)\kappa r\}.$$

We obtain a real function for ψ by putting

$$\psi=(\alpha+\beta i)e^{mti}J_0\{(1-i)\kappa r\}+(\alpha-\beta i)e^{-mti}J_0\{(1+i)\kappa r\}, \qquad 73$$

α, β, m being any real constants. Suppose the velocity is prescribed to be $a\omega\sin mt$ when $r=a$; then

$$\begin{aligned}a\omega\sin mt&=-\left(\frac{\partial\psi}{\partial r}\right)_{r=a}\\&=\kappa(1-i)(\alpha+\beta i)e^{mti}J_1\{(1-i)\kappa a\}\\&\quad+\kappa(1+i)(\alpha-\beta i)e^{-mti}J_1\{(1+i)\kappa a\},\end{aligned}$$

and therefore

$$\kappa(1-i)(\alpha+\beta i)J_1\{(1-i)\kappa a\}=-\frac{ia\omega}{2},$$

so that

$$\psi=\frac{a\omega(1-i)J_0\{(1-i)\kappa r\}}{4\kappa J_1\{(1-i)\kappa a\}}e^{mti}+\frac{a\omega(1+i)J_0\{(1+i)\kappa r\}}{4\kappa J_1\{(1+i)\kappa a\}}e^{-mti}. \qquad 74$$

In order to obtain this explicitly in a real form, let us write

$$\frac{(1-i)J_0\{(1-i)\kappa r\}}{\kappa J_1\{(1-i)\kappa a\}}=P+Qi,$$

P and Q being real functions: then

$$\begin{aligned}\psi &= \tfrac{1}{4}a\omega\,\{(P+Qi)(\cos mt + i\sin mt) \\ &\qquad\qquad + (P-Qi)(\cos mt - i\sin mt)\} \\ &= \tfrac{1}{2}a\omega\,(P\cos mt - Q\sin mt).\end{aligned} \tag{75}$$

The boundary condition may be realised by supposing the liquid to fill the interior of an infinite cylinder of radius a, which is constrained to move with angular velocity $\omega \sin mt$ about its axis, carrying with it the particles of liquid which are in contact with it.

(This example is taken from the paper set in the Mathematical Tripos, Wednesday afternoon, Jan. 3, 1883.)

A very important application of the theory is contained in Stokes's memoir "On the effect of the internal friction of fluids on the motion of pendulums" (*Camb. Phil. Trans.*, vol. IX.): for the details of the investigation the reader should consult the original paper, but we shall endeavour to give an outline of the analysis.

The practical problem is that of taking into account the viscosity of the air in considering the small oscillations, under the action of gravity, of a cylindrical pendulum. In order to simplify the analysis, we begin by supposing that we have an infinite cylinder of radius a, surrounded by viscous liquid of density ρ, also extending to infinity; and we proceed to construct a possible state of two-dimensional motion in which the cylinder moves to and fro along the initial line $\theta = 0$ in such a way that its velocity V at any instant is expressed by the formula

$$V = ce^{2\nu nti} + c_0 e^{-2\nu n_0 ti}, \tag{76}$$

where $\nu = \mu/\rho$, μ being the coefficient of viscosity; n, n_0 are conjugate complex constants, and c, c_0 are conjugate complex constants of small absolute value.

The current function ψ must vanish at infinity, and satisfy the equation

$$\nabla^2\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\psi = 0, \tag{77}$$

and, in addition, the boundary conditions

$$\frac{\partial\psi}{\partial\theta} = Va\cos\theta, \quad \frac{\partial\psi}{\partial r} = V\sin\theta, \tag{78}$$

when $r = a$.

Now if we assume

$$\psi = \left[e^{2\nu nti}\left\{\frac{A}{r} + B\chi(r)\right\} + e^{-2\nu n_0 ti}\left\{\frac{A_0}{r} + B_0\chi_0(r)\right\}\right]\sin\theta \qquad 79$$

part of this expression, namely the sum of the first and third terms, satisfies the equation

$$\nabla^2\psi = 0,$$

and the remaining part satisfies

$$\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\psi = 0,$$

provided the functions χ, χ_0 are chosen so that

$$\frac{d^2\chi}{dr^2} + \frac{1}{r}\frac{d\chi}{dr} - \left(2in + \frac{1}{r^2}\right)\chi = 0,$$

$$\frac{d^2\chi_0}{dr^2} + \frac{1}{r}\frac{d\chi_0}{dr} - \left(-2in_0 + \frac{1}{r^2}\right)\chi_0 = 0.$$

These equations are satisfied by

$$\chi = K_1\{(1+i)\sqrt{nr}\},$$
$$\chi_0 = K_1\{(1-i)\sqrt{n_0 r}\}.$$

Put

$$(1+i)\sqrt{n} = \lambda,$$
$$(1-i)\sqrt{n_0} = \lambda_0;$$

then

$$\left.\begin{aligned}\chi &= K_1(\lambda r) = P + iQ,\\ \chi_0 &= K_1(\lambda_0 r) = P - iQ\end{aligned}\right\}, \qquad 80$$

where P and Q are real functions of r.

The boundary conditions are satisfied if

$$\frac{A}{a} + B\chi(a) = ca,$$

$$-\frac{A}{a^2} + B\chi'(a) = c;$$

whence

$$\left.\begin{aligned}A &= \frac{ca^2\{a\chi'(a) - \chi(a)\}}{\chi(a) + a\chi'(a)},\\ B &= \frac{2ca}{\chi(a) + a\chi'(a)}\end{aligned}\right\}, \qquad 81$$

and A_0, B_0 are obtained from these by changing i into $-i$.

The equations 79, 80, 81 may be regarded as giving the motion of the fluid when the cylinder is *constrained* to move according to the law expressed by 76.

By proceeding as in Basset's *Hydrodynamics*, II. 280 it may be shown that the resistance to the motion of the cylinder, arising from the viscosity of the surrounding liquid, amounts, per unit length of the cylinder, to

$$Z = 2\pi\rho\nu n i a\,(Le^{2\pi n t i} - L_0 e^{-2\pi n_0 t i}), \tag{82}$$

where

$$\begin{aligned} L &= \frac{A}{a} - B\chi(a) \\ &= \frac{ca\{a\chi'(a) - 3\chi(a)\}}{\chi(a) + a\chi'(a)}, \end{aligned} \tag{83}$$

and L_0 is conjugate to L.

Let σ be the density of the cylinder: then the force which must act at time t upon each unit length of it, in order to maintain the prescribed motion, is

$$\begin{aligned} F &= \pi\sigma a^2 \frac{dV}{dt} + Z \\ &= 2\pi i \nu a^2 (Ne^{2\pi n t i} - N_0 e^{-2\pi n_0 t i}), \end{aligned} \tag{84}$$

with

$$\left.\begin{aligned} N &= nc\left[\sigma + \frac{\rho\{a\chi'(a) - 3\chi(a)\}}{\chi(a) + a\chi'(a)}\right], \\ N_0 &= \text{the conjugate quantity.} \end{aligned}\right\} \tag{85}$$

Now let us suppose that we have a pendulum consisting of a heavy cylindrical bob suspended by a fine wire and making small oscillations in air under the action of gravity. We shall assume that when the amplitude of the oscillation is sufficiently small, and the period sufficiently great, the motion will be approximately of the same type as that which has just been worked out for an infinite cylinder; so that if ξ is the horizontal displacement of the bob at time t from its mean position, we shall have

$$\dot{\xi} = V = ce^{2\pi n t i} + c_0 e^{-2\pi n_0 t i}. \tag{86}$$

The force arising from gravity which acts upon the bob is, per unit of length, and to the first order of small quantities,

$$-\pi(\sigma - \rho)\,a^2 \cdot \frac{g}{l}\,\xi,$$

where l is the distance of the centre of mass of the pendulum from the point of suspension.

Equating this to the value of F given above, we have the conditional equation

$$2i\nu l\,(Ne^{2\nu nti} - N_0 e^{-2\nu n_0 ti}) + (\sigma - \rho)\,g\xi = 0, \qquad 87$$

which must hold at every instant, and may therefore be differentiated with regard to the time. Doing this, and substituting for $\dot{\xi}$ its value in terms of the time, we obtain

$$\{-4n\nu^2 lN + (\sigma - \rho)\,gc\}\,e^{2\nu nti} + \{-4n_0\nu^2 lN_0 + (\sigma - \rho)\,gc_0\}\,e^{-2\nu n_0 ti} = 0,$$

which is satisfied identically if we put

$$(\sigma - \rho)\,gc = 4n\nu^2 lN,$$

or, which is the same thing,

$$\frac{(\sigma - \rho)\,g}{l} = 4\left(\sigma + \frac{a\chi'(a) - 3\chi(a)}{\chi(a) + a\chi'(a)}\,\rho\right)\nu^2 n^2. \qquad 88$$

This, with $\chi(a)$ defined by 80 above, is an equation to find n which must be solved by approximation: since the motion is actually retarded, the proper value of n must have a positive imaginary part. As might be expected, when ρ is very small in comparison with σ,

$$4\nu^2 n^2 = g/l$$

approximately.

The constants c and c_0 are determined by the initial values of ξ and $\dot{\xi}$, together with the equations 86 and 87.

CHAPTER XII.

STEADY FLOW OF ELECTRICITY OR OF HEAT IN UNIFORM ISOTROPIC MEDIA.

CHAPTER VII. above, which deals with Fourier-Bessel Expansions, contains all that is required for the application of Bessel Functions to problems regarding the distribution of potential; but it may be advisable to supplement that theoretical discussion by a few examples fully worked out. We take here a few cases of electric flow of some physical interest. Other problems with notes as to their solution in certain cases will be found in the collection of Examples at the end of the book. In the discussions in this chapter we speak of the flow as electric; but the problems solved may be regarded as problems in the theory of the steady flux of heat or incompressible fluid moving irrotationally, or even of the distribution of potential and force in an electrostatic field. The method of translation is well understood. The potential in the flux of electricity becomes the temperature in the thermal analogue, while the conductivities and strength of source (or sink) involve no change of nomenclature; the potential in the flux theory and that in the electrostatic theory coincide, the sources and sinks in the former become positive and negative charges in the latter, while specific inductive capacity takes the place of conductivity.

If V be the potential, then in all the problems here considered the differential equation which holds throughout the medium is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \qquad 1$$

or in cylindrical coordinates r, θ, z,

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0. \qquad 2$$

At the surface of separation of two media of different conductivities k_1, k_2 the condition which holds is

$$k_1 \frac{\partial V_1}{\partial n_1} + k_2 \frac{\partial V_2}{\partial n_2} = 0, \qquad 3$$

where n_1, n_2 denote normals drawn from a point of the surface into the respective media, and V_1, V_2 are the potentials in the two media infinitely near that point. If one of the media is an insulator, so that say $k_2 = 0$, the equation of condition is

$$\frac{\partial V}{\partial n} = 0. \qquad 4$$

Let us define a source or sink as a place where electricity is led into or drawn off from the medium, and consider the electricity delivered or drawn off uniformly over a small spherical electrode of perfectly conducting substance (of radius r) buried in the medium at a distance great in comparison with r from any part of the bounding surface. Let it be kept at potential V, and deliver or withdraw a total quantity S per unit of time, then since $V = \text{constant}/r$,

$$\frac{V}{S} = \frac{1}{4\pi kr}. \qquad 5$$

The quantity on the right is half the resistance between a source and a sink thus buried in the medium and kept at a difference of potential $2V$.

If the electrode is on the surface (supposed of continuous curvature) of the medium the electrode must be considered as a hemisphere, and the resistance will be double the former amount. In this case

$$\frac{V}{S} = \frac{1}{2\pi kr}. \qquad 6$$

When r is made infinitely small we must have rV finite, and therefore in the two cases just specified

$$\left.\begin{aligned} \text{Lt}\, rV &= \frac{S}{4\pi k}, \\ \text{Lt}\, rV &= \frac{S}{2\pi k} \end{aligned}\right\}. \qquad 7$$

Equations 1, 2, 3, and 7 are the conditions to be fulfilled in the problems which we now proceed to give examples of. Those we here choose are taken from a very instructive paper

by Weber ("Ueber Bessel'sche Functionen und ihre Anwendung auf die Theorie der elektrischen Ströme," *Crelle*, Bd. 75, 1873), and are given with only some changes in notation to suit that adopted in the present treatise, and the addition of some explanatory analysis.

We shall prove first the following proposition. If V be the potential due to a circular disk of radius r_1 on which there is a charge of electricity in equilibrium unaffected by the action of electricity external to the disk, then if z be taken along the axis of the disk, and the origin at the centre,

$$V = \frac{2c}{\pi} \int_0^\infty e^{\mp\lambda z} \sin(\lambda r_1) J_0(\lambda r) \frac{d\lambda}{\lambda}, \tag{8}$$

where the upper sign is to be taken for positive values of z and the lower for negative values, and c is the potential at the disk.

In the first place this expression for V satisfies 2; if then we can prove that it reduces to a constant when $z = 0$, and gives the proper value of the electric density we shall have verified the solution. By 46, p. 18 above, if $\epsilon > 0$,

$$\begin{aligned} &\int_0^\infty e^{-\epsilon\lambda} \sin(\lambda r_1) J_0(\lambda r) \frac{d\lambda}{\lambda} \\ &\qquad = \frac{1}{\pi} \int_0^\infty e^{-\epsilon\lambda} \sin(\lambda r_1) \left\{ \int_0^\pi \cos(\lambda r \sin\theta)\, d\theta \right\} \frac{d\lambda}{\lambda} \\ &\qquad = \frac{1}{\pi} \int_0^\pi d\theta \int_0^\infty e^{-\epsilon\lambda} \sin\lambda r_1 \cos(\lambda r \sin\theta) \frac{d\lambda}{\lambda}, \end{aligned} \tag{9}$$

since changing the order of integration is permissible here.

Consider the integral

$$\int_0^\infty e^{-\epsilon\lambda} \sin(\lambda r_1) \cos(\lambda r \sin\theta) \frac{d\lambda}{\lambda}.$$

This can be written

$$\frac{1}{2} \int_0^\infty e^{-\epsilon\lambda} \left[\sin\{\lambda(r_1 + r\sin\theta)\} + \sin\{\lambda(r_1 - r\sin\theta)\}\right] \frac{d\lambda}{\lambda}.$$

But we know that if $a > 0$

$$\int_0^\infty e^{-ax} \sin x\, dx = \frac{1}{a^2+1}.$$

Multiplying this equation by da and integrating both sides from $a=\epsilon$ to $a=\infty$ ($\epsilon>0$), we obtain

$$\int_0^\infty e^{-\epsilon x}\sin x\,\frac{dx}{x}=\frac{\pi}{2}-\tan^{-1}\epsilon.$$

Thus the integral considered has the value

$$\frac{\pi}{2}-\frac{1}{2}\tan^{-1}\frac{\epsilon}{r_1+r\sin\theta}-\frac{1}{2}\tan^{-}\frac{\epsilon}{r_1-r\sin\theta},$$

and

$$\int_0^\pi d\theta\int_0^\infty e^{-\epsilon\lambda}\sin(\lambda r_1)\cos(\lambda r\sin\theta)\frac{d\lambda}{\lambda}$$

$$=\frac{\pi^2}{2}-\frac{1}{2}\int_0^\pi d\theta\tan^{-1}\frac{\epsilon}{r_1+r\sin\theta}-\frac{1}{2}\int_0^\pi d\theta\tan^{-1}\frac{\epsilon}{r_1-r\sin\theta}. \qquad 10$$

The integral on the left is convergent for all positive values of ϵ including 0. Hence if we evaluate the equivalent expression on the right for a very small positive value of ϵ we shall obtain the value of the integral on the right of 8 when $z=0$. In doing this there is no difficulty if $r_1>r$; but if $r_1<r$, the element of the last integral in 10, for which $\theta=\sin^{-1}r_1/r$, is $\frac{\pi}{2}\,d\theta$, and the integral requires discussion.

The first of the two integrals on the right vanishes if ϵ be very small. The second also vanishes when $r_1>r$, so that the integral sought is in that case $\pi^2/2$. Now

$$\int_0^\pi d\theta\tan^{-1}\frac{\epsilon}{r_1-r\sin\theta}=2\int_0^{\frac{\pi}{2}}d\theta\left(\frac{\pi}{2}-\tan^{-1}\frac{r_1-r\sin\theta}{\epsilon}\right)$$

$$=\frac{\pi^2}{2}-2\int_0^{\sin^{-1}\frac{r_1}{r}}\tan^{-1}\frac{r_1-r\sin\theta}{\epsilon}\,d\theta-2\int_{\sin^{-1}\frac{r_1}{r}}^{\frac{\pi}{2}}\tan^{-1}\frac{r_1-r\sin\theta}{\epsilon}\,d\theta.$$

Each element of the first integral just written down on the right is $\frac{1}{2}\pi d\theta$ except just when $\theta=\sin^{-1}(r_1/r)$, when it vanishes, since ϵ is not zero. Similarly except just at the beginning each element of the second is $-\frac{1}{2}\pi d\theta$. Hence for $r>r_1$

$$-\frac{1}{2}\int_0^\pi d\theta\tan^{-1}\frac{\epsilon}{r_1-r\sin\theta}=\pi\sin^{-1}\frac{r_1}{r}-\frac{\pi^2}{2},$$

and we have finally for $\epsilon = 0$

$$\left.\begin{aligned} \frac{2c}{\pi}\int_0^\infty \sin(\lambda r_1)\, J_0(\lambda r)\frac{d\lambda}{\lambda} &= c, \text{ if } r < r_1 \\ &= \frac{2c}{\pi}\sin^{-1}\frac{r_1}{r}, \text{ if } r > r_1; \end{aligned}\right\} \qquad 11$$

when $r_1 = r$, the two results coincide.

Thus the expression

$$V = \frac{2c}{\pi}\int_0^\infty e^{\mp\lambda z}\sin(\lambda r_1)\, J_0(\lambda r)\frac{d\lambda}{\lambda} \qquad 12$$

satisfies the differential equation, gives a constant potential at every point of the disk of radius r_1, and is, as well as $\partial V/\partial z$, continuous when $z = 0$, for all values of r.

Lastly to find the distribution, we have for $z = +0$

$$-\frac{1}{4\pi}\frac{\partial V}{\partial z} = \frac{c}{2\pi^2}\int_0^\infty \sin(\lambda r_1)\, J_0(\lambda r)\, d\lambda = \frac{c}{2\pi^2}\frac{1}{\sqrt{r_1^2 - r^2}} \qquad 13$$

by 151, p. 73 above. Or the whole density, taking the two faces of the disk together, is $c/\pi^2\sqrt{r_1^2 - r^2}$. This is a result which can be otherwise obtained. Hence the solution is completely verified.

We can now convert this result into the solution of a problem in the flow of electricity. Let us suppose that the electrode supplying electricity is the disk we have just imagined, and let it be composed of perfectly conducting material, and be immersed in an unlimited medium of conductivity k. Then to a constant the potential at any point of the electrode is

$$V = \frac{2c}{\pi}\int \sin(\lambda r_1)\, J_0(\lambda r)\frac{d\lambda}{\lambda}. \qquad 14$$

The sink or sinks may be supposed at a very great distance so that they do not disturb the flow in the neighbourhood of this disk-shaped source.

The rate of flow from the disk to the medium is $-k\partial V/\partial z$ per unit of area at each point of the electrode, and is of course in the direction of the normal. At the edge by 13 the flow will be infinite if the disk is a very thin oblate ellipsoid of revolution, as it is here supposed to be; but in this, and in any actual case, the total flow from the vicinity of the edge can obviously be made

as small as we please in comparison with the total flow elsewhere by increasing the radius of the disk.

The total flow from the disk to the medium is thus

$$S = -2k\int_0^{2\pi}\int_0^{r_1}\frac{\partial V}{\partial z}\,r\,dr\,d\phi.$$

Putting in this for $\partial V/\partial z$ its value we get

$$S = -\frac{4ck}{\pi}\int_0^{2\pi}\{\sqrt{r_1^2 - r^2}\}_0^{r_1}\,d\phi$$

$$= 8ckr_1.$$

Thus the amount supplied by each side of the disk per unit time is $4ckr_1$, and we have

$$c = \frac{S}{8kr_1}. \qquad 15$$

If the disk is laid on the bounding surface of a conductor the flow will take place only from one face to the conducting mass, and S has only half of its value in the other case. Then

$$c = \frac{S}{4kr_1}. \qquad 16$$

In this case the condition $\partial V/\partial n = 0$ holds all over the surface except at the disk-electrode, and of course 2 holds within the conductor. At any point of the disk distant r from the centre

$$-\frac{\partial V}{\partial z} = \frac{2c}{\pi}\frac{1}{\sqrt{r_1^2 - r^2}} = \frac{S}{2\pi kr_1}\frac{1}{\sqrt{r_1^2 - r^2}}. \qquad 17$$

We can now find the resistance of the conducting mass between two such conducting electrodes, a source and a sink, placed anywhere on the surface at such a distance apart that the stream-lines from or to either of them are not in its neighbourhood disturbed by the position of the other. The whole current up to the disk by which the current enters is S, and we have seen that c is the potential of that disk. For distinction let the potentials of the source and sink disks be denoted by c_1, c_2; then if R be the resistance between them

$$R = \frac{c_1 - c_2}{S}.$$

If the wires leading the current up to and away from the electrodes have resistances ρ_1, ρ_2, and have their farther extremities

(at the generator or battery) at potentials V_1, V_2, the falls of potential along the inleading electrode, and along the outgoing are

$$V_1 - c_1 = S\rho_1, \quad c_2 - V_2 = S\rho_2,$$

so that
$$V_1 - V_2 - (c_1 - c_2) = S(\rho_1 + \rho_2),$$

and
$$R = \frac{1}{S}\{V_1 - V_2 - S(\rho_1 + \rho_2)\}. \qquad 18$$

Another expression for the resistance can be found as follows. We have seen that the potential at the source-disk is c_1, also that for conduction from one side of the disk

$$S = 4c_1kr_1.$$

For the sink-disk the outward current in like manner is

$$S = -4c_2kr_2.$$

Hence
$$S = 2k(c_1r_1 - c_2r_2).$$

But also
$$R = \frac{c_1 - c_2}{S} = \frac{c_1 - c_2}{2k(c_1r_1 - c_2r_2)},$$

and $c_1r_1 = -c_2r_2$, so that

$$R = \frac{r_1 + r_2}{4kr_1r_2} = \frac{1}{4kr_1} + \frac{1}{4kr_2}. \qquad 19$$

From the latter form of the result we infer that $1/4kr_1$ is the part of the resistance due to the first disk, $1/4kr_2$ the part due to the second. This result is of great importance, for it gives a means of calculating an inferior limit to the correction to be made on the resistance of a cylindrical wire in consequence of its being joined to a large mass of metal.

From this problem we can proceed to another which is identical with that of Nobili's rings solved first by Riemann. An infinite conductor is bounded by two parallel planes $z = \pm a$, and two disk electrodes are applied to these planes, so that their centres lie in the axis of z. It is required to find the potential at each point of the conductor and the resistance between the electrodes. From the distribution of potential the stream-lines can of course be found also.

The solution must fulfil the following conditions:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0, \text{ for } -a < z < +a,$$

$$\frac{\partial V}{\partial z} = 0, \qquad \text{for } z = \pm a, r > r_1,$$

$$+\frac{\partial V}{\partial z} = \frac{S}{2\pi k r_1 \sqrt{r_1^2 - r^2}}, \text{ for } z = \pm a, \ r < r_1.$$

According to the last condition the current is supposed to flow along the axis in the direction of z decreasing.

The first condition is satisfied by assuming

$$V = \int_0^\infty \{\phi(\lambda) e^{\lambda z} + \psi(\lambda) e^{-\lambda z}\} J_0(\lambda r) \, d\lambda \qquad 20$$

where $\phi(\lambda)$, $\psi(\lambda)$ are arbitrary functions of λ which render the integral convergent and fulfil the other necessary conditions.

Without loss of generality V may be supposed zero when $z = 0$, and hence we must put $\phi(\lambda) = -\psi(\lambda)$. Thus 20 becomes

$$V = \int_0^\infty 2\phi(\lambda) \sinh(\lambda z) J_0(\lambda r) \, d\lambda. \qquad 21$$

With regard to the other two conditions, by 150, 151, p. 73, above,

$$\int_0^\infty \sin(\lambda r_1) J_0(\lambda r) \, d\lambda = 0, \text{ when } r > r_1$$

$$= \frac{1}{\sqrt{r_1^2 - r^2}}, \text{ when } r < r_1.$$

Hence if we take

$$2\phi(\lambda) \cosh(\lambda a) . \lambda = \frac{S}{2\pi k r_1} \sin(\lambda r_1),$$

or

$$\phi(\lambda) = \frac{S}{4\pi k r_1} \frac{\sin \lambda r_1}{\cosh(\lambda a)} \frac{1}{\lambda} \qquad 22$$

both conditions will be satisfied. The solution of the problem is therefore

$$V = \frac{S}{2\pi k r_1} \int_0^\infty \frac{\sinh(\lambda z)}{\cosh(\lambda a)} \sin(\lambda r_1) J_0(\lambda r) \frac{d\lambda}{\lambda}. \qquad 23$$

From this we can easily obtain an approximation to the resistance R between the electrodes. For we have from 23

$$c_1 - c_2 = \frac{S}{\pi k r_1}\int_0^\infty \tanh(\lambda a) J_0(\lambda r)\sin(\lambda r_1)\frac{d\lambda}{\lambda}$$

$$= \frac{S}{\pi k r_1}\int_0^\infty \left(1 - \frac{2e^{-2\lambda a}}{1+e^{-2\lambda a}}\right) J_0(\lambda r)\sin\lambda r_1 \frac{d\lambda}{\lambda}. \qquad 24$$

If the second term in brackets be neglected in the last expression, we have to a first approximation, since by 11 the integral is equal to $\pi/2$,

$$R = \frac{c_1 - c_2}{S} = \frac{1}{2kr_1}.$$

This of course could have been obtained at once from 19 by simply putting $r_1 = r_2$. To obtain a nearer approximation the expression on the right in 24 may be expanded in powers of r and r_1. If terms of the order r_1^2/a^2 and upwards be neglected, the result is

$$R = \frac{1}{2kr_1} - \frac{\log 2}{\pi k a}. \qquad 25$$

If the electrodes are extremely small we may put λr_1 for $\sin \lambda r_1$ and we obtain from 23

$$V = \frac{S}{2\pi k}\int_0^\infty \frac{\sinh(\lambda z)}{\cosh(\lambda a)} J_0(\lambda r)\, d\lambda. \qquad 26$$

This expression applies to the space between the two planes $z = \pm a$. Hence expanding by Fourier's method we obtain

$$\frac{\sinh(\lambda z)}{\cosh(\lambda a)} = \frac{2\lambda}{a}\sum_1^\infty \frac{1}{\lambda^2 + \left(\frac{n\pi}{2a}\right)^2}\sin\frac{n\pi}{2}\sin\frac{n\pi z}{2a}. \qquad 27$$

Hence

$$V = \frac{S}{\pi k a}\sum_1^\infty \sin\frac{n\pi}{2}\sin\frac{n\pi z}{2a}\int_0^\infty \frac{J_0(\lambda r)\,\lambda\, d\lambda}{\lambda^2 + \left(\frac{n\pi}{2a}\right)^2}. \qquad 28$$

Now it will be proved, p. 200 below, that if x be positive

$$J_0(x) = \frac{2}{\pi}\int_1^\infty \frac{\sin(\xi x)\, d\xi}{\sqrt{\xi^2 - 1}}.$$

Hence putting κ^2 for $(n\pi/2a)^2$ we have

$$\int^\infty \frac{J_0(\lambda r)\,\lambda\, d\lambda}{\kappa^2 + \lambda^2} = \frac{2}{\pi}\int_1^\infty \frac{d\xi}{\sqrt{\xi^2 - 1}}\int_0^\infty \frac{\lambda \sin(\xi r \lambda)\, d\lambda}{\kappa^2 + \lambda^2}.$$

But it can be shown that according as $\xi r >$ or < 0,

$$\int_0^\infty \frac{\lambda \sin(\xi r\lambda)\,d\lambda}{\kappa^2+\lambda^2} = \pm\frac{\pi}{2}e^{\mp\kappa\xi r}. \tag{29}$$

Thus since ξ and r are both here positive

$$\int_0^\infty \frac{J_0(\lambda r)\,\lambda d\lambda}{\kappa^2+\lambda^2} = \int_1^\infty \frac{e^{-\kappa\xi r}\,d\xi}{\sqrt{\xi^2-1}} = \int_1^\infty \frac{e^{-\kappa r\lambda}\,d\lambda}{\sqrt{\lambda^2-1}}. \tag{30}$$

Substituting in 28 we obtain

$$V = \frac{S}{\pi ka}\sum_1^\infty \sin\frac{n\pi}{2}\sin\frac{n\pi z}{2a}\int_1^\infty \frac{e^{-\kappa r\lambda}\,d\lambda}{\sqrt{\lambda^2-1}}, \tag{31}$$

which agrees with the solution of the problem given by Riemann (*Werke*, p. 58, or *Pogg. Ann.* Bd. 95, March, 1855).

If the conducting mass instead of being infinite be a circular cylinder of axis z and radius c, bounded by non-conducting matter, the problem becomes more complicated. To solve it in this case a part V' must be added to V fulfilling the following conditions:

(1) $$\frac{\partial^2 V'}{\partial r^2} + \frac{1}{r}\frac{\partial V'}{\partial r} + \frac{\partial^2 V'}{\partial z^2} = 0, \text{ for } r < c, \ -a < z < +a,$$

(2) $$\frac{\partial V'}{\partial z} = 0, \text{ for } z = \pm a,$$

(3) $$\frac{\partial V'}{\partial r} + \frac{\partial V}{\partial r} = 0, \text{ for } r = c, \ -a < z < +a.$$

If we write

$$\left.\begin{aligned} L_1(r) &= \int_1^\infty \frac{e^{-\kappa r\lambda}\,d\lambda}{\sqrt{\lambda^2-1}}, \quad M_1(r) = \int_1^\infty \frac{e^{-\kappa r\lambda}\,\lambda d\lambda}{\sqrt{\lambda^2-1}} \\ L_2(r) &= \int_{-1}^{+1} \frac{e^{-\kappa r\lambda}\,d\lambda}{\sqrt{1-\lambda^2}}, \quad M_2(r) = \int_{-1}^{+1} \frac{e^{-\kappa r\lambda}\,\lambda d\lambda}{\sqrt{1-\lambda^2}} \end{aligned}\right\} \tag{32}$$

and denote by $L_1(c)$, $L_2(c)$, &c. the same quantities with c substituted for r, the conditions stated are found to be fulfilled by

$$V' = -\frac{S}{\pi ka}\sum_1^\infty \sin\frac{n\pi}{2}\sin\frac{n\pi z}{2a}\frac{M_1(c)}{M_2(c)}L_2(r). \tag{33}$$

For consider the series

$$b_1 \sin\frac{\pi z}{2a} + b_3 \sin\frac{3\pi z}{2a} + \ldots\ldots \tag{34}$$

The coefficients b_1, b_3, ..., which are functions of r, can be so taken as to fulfil the differential equation in the medium. Thus the first three conditions are fulfilled.

By the general differential equation we have

$$\frac{\partial^2 b_n}{\partial r^2}+\frac{1}{r}\frac{\partial b_n}{\partial r}-\kappa^2 b_n=0 \qquad 35$$

of which there are two known solutions

$$\int_1^\infty \frac{e^{-\kappa r\lambda}d\lambda}{\sqrt{\lambda^2-1}},\quad \int_{-1}^{+1}\frac{e^{-\kappa r\lambda}d\lambda}{\sqrt{1-\lambda^2}}.$$

If the first is expanded in powers of r and integrated it is found that it becomes infinite for $r=0$. We therefore take as the solution of 35

$$b_n=\beta_n\int_{-1}^{+1}\frac{e^{-\kappa r\lambda}d\lambda}{\sqrt{1-\lambda^2}},$$

where β_n is a constant to be determined by the remaining equation of condition, (3). We obtain in the notation of 32

$$\beta_n=\frac{-S}{\pi ka}\frac{M_1(c)}{M_2(c)}\sin\frac{n\pi}{2} \qquad 36$$

and the total potential at any point is

$$V+V'=\frac{S}{\pi ka}\sum_1^\infty \sin\frac{n\pi}{2}\sin\frac{n\pi z}{2a}\frac{M_2(c)L_1(r)-M_1(c)L_2(r)}{M_2(c)}. \qquad 37$$

If in 37 we were to put $r=0$, $z=\pm a$, and could evaluate the integrals we should obtain c_1-c_2, the difference of potential now existing for the given total flow S. This divided by S would give the resistance.

When $r=0$, $L_2(r)=\pi$, so that the change in the resistance due to the limitation of the flow to the finite cylinder is

$$-\frac{2}{ka}\sum_n\frac{M_1(c)}{M_2(c)}=\frac{2}{ka}e^{-\frac{\pi c}{a}}$$

if $\frac{a}{c}$ be very small. Hence the resistance is approximately

$$R=\frac{1}{2kr_1}-\frac{\log 2}{\pi ka}+\frac{2}{ka}e^{-\frac{\pi c}{a}}. \qquad 38$$

We now pass on to another problem also considered by Weber. A plane metal plate which may be regarded as of infinite extent, is separated from a conductor of relatively smaller conductivity by a thin stratum of slightly conducting material. For example, this may be a film of gas separating an electrode of metal from a con-

ducting liquid as in cases of polarization in cells. We shall calculate the resistance for the case in which the electrode is small and is applied at a point within the conducting mass. Take the axis of z along the line through the point electrode perpendicular to the metal plate, and the origin on the surface of the conductor close to the plate. Thus the point electrode is applied at the point $z = a$, $r = 0$.

We further suppose that there is a difference of potential w between the surface of the conductor and the metal plate on the other side of the film. This will give a slope of potential through the film of amount w/δ if δ be the film thickness. If the conductivity of the film be k_1 the resistance for unit of area will be δ/k_1, and thus the flow per unit of area across the film is wk_1/δ. This must be equal to the rate at which electricity is conducted up to the surface of the conductor from within, which is $k\partial V/\partial z$. Thus if w be the *positive* difference between the plate and the conductor surface, the condition holds when $z = 0$,

$$-h\frac{\partial V}{\partial z} + w = 0,$$

where $$h = \delta k/k_1.$$

Let ρ, ρ', be the distances of any point z, r from the electrode and from its image in the surface respectively. Then the differential equation and the other conditions laid down are satisfied by

$$V = \frac{S}{4\pi k}\left(\frac{1}{\rho} - \frac{1}{\rho'}\right) + w, \qquad 39$$

provided that w fulfils the equations

$$-h\frac{\partial V}{\partial z} + w = 0$$

at the surface, and

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} = 0, \qquad 40$$

throughout the conductor. The first term on the right is the solution we should have had if the film had not existed, the second is the increased potential at each point in consequence of the rise in crossing the film from the plate.

A value of w which satisfies 40 is given by

$$w = \int_0^\infty e^{-\lambda z}\phi(\lambda)\, J_0(\lambda r)\, d\lambda, \qquad 41$$

where $\phi(\lambda)$ is an arbitrary function of λ to be determined. Now since

$$\rho^2 = (z-a)^2 + r^2, \quad \rho'^2 = (z+a)^2 + r^2,$$

$$\frac{\partial V}{\partial z} = -\frac{S}{4\pi k}\frac{z-a}{\rho^3} + \frac{S}{4\pi k}\frac{z+a}{\rho'^3} + \frac{\partial w}{\partial z}$$

$$= \frac{S}{2\pi k}\frac{a}{(a^2+r^2)^{\frac{3}{2}}} + \frac{\partial w}{\partial z},$$

when $z = 0$. Hence the surface condition becomes having regard to 41,

$$\frac{hS}{2\pi k}\frac{a}{(a^2+r^2)^{\frac{3}{2}}} - \int_0^\infty (1+h\lambda)\,\phi(\lambda)\,J_0(\lambda r)\,d\lambda = 0. \qquad 42$$

But differentiating with respect to a the equation

$$\int_0^\infty e^{-a\lambda} J_0(\lambda r)\,d\lambda = \frac{1}{(a^2+r^2)^{\frac{1}{2}}}$$

we get

$$\int_0^\infty e^{-a\lambda} J_0(\lambda r)\,\lambda d\lambda = \frac{a}{(a^2+r^2)^{\frac{3}{2}}}. \qquad 43$$

This substituted in 42 gives

$$\phi(\lambda) = \frac{hS}{2\pi k}\frac{\lambda e^{-a\lambda}}{1+h\lambda}.$$

Hence $$w = \frac{hS}{2\pi k}\int_0^\infty e^{-\lambda(z+a)}\frac{J_0(\lambda r)\,\lambda d\lambda}{1+h\lambda}.$$

Now $$e^{\frac{z+a}{h}}\int_{\frac{z+a}{h}}^\infty e^{-(1+h\lambda)t}\,dt = \frac{e^{-\lambda(z+a)}}{1+h\lambda},$$

so that we have

$$w = \frac{hS}{2\pi k} e^{\frac{z+a}{h}} \int_{\frac{z+a}{h}}^\infty dt\, e^{-t} \int_0^\infty e^{-h\lambda t} J_0(\lambda r)\,\lambda d\lambda$$

$$= \frac{S}{2\pi hk} e^{\frac{z+a}{h}} \int_{\frac{z+a}{h}}^\infty \frac{te^{-t}dt}{\left(t^2+\frac{r^2}{h^2}\right)^{\frac{3}{2}}}, \text{ by } 43$$

$$= \frac{S}{2\pi k\rho'} - \frac{S}{2\pi kh} e^{\frac{z+a}{h}} \int_{\frac{z+a}{h}}^\infty \frac{e^{-t}dt}{\left(t^2+\frac{r^2}{h^2}\right)^{\frac{1}{2}}} \qquad 44$$

by integration by parts. Thus we obtain for the potential at any point z, r

$$V=\frac{S}{4\pi k}\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)-\frac{S}{2\pi kh}e^{\frac{z+a}{h}}\int_{\frac{z+a}{h}}^{\infty}\frac{e^{-t}dt}{\left(t^2+\frac{r^2}{h^2}\right)^{\frac{1}{2}}}$$

Take a new variable ζ given by

$$ht=\zeta+z+a,$$

and the solution becomes

$$V=\frac{S}{4\pi k}\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)-\frac{S}{2\pi kh}\int_0^{\infty}\frac{e^{-\frac{\zeta}{h}}d\zeta}{\sqrt{(\zeta+z+a)^2+r^2}}. \qquad 45$$

The meaning of this solution is that the introduction of the non-conducting film renders the distribution of potential that which would exist for the same total flow S, were there a combination of two equal positive sources of strength $S/4\pi k$, at the electrode and its image, with a linear source extending along the axis of z from the image to $-\infty$, and of intensity

$$-\frac{S}{2\pi kh}e^{-\frac{\zeta}{h}}$$

per unit of length, at distance ζ from the point $-(z+a)$.

If the conducting mass be of small thickness then nearly enough $\rho=\rho'=r$, and $z+a=a$. Thus we obtain

$$V=\frac{S}{2\pi k}\left\{\frac{1}{r}-\int_0^{\infty}\frac{e^{-t}dt}{\sqrt{h^2t^2+r^2}}\right\} \qquad 46$$

if as we suppose $(z+a)/h$ may be neglected.

If h/r be small we can expand $(h^2t^2+r^2)^{-\frac{1}{2}}$ in ascending powers of t by the binomial theorem and integrate term by term. We thus get

$$\int_0^{\infty}\frac{e^{-t}dt}{\sqrt{h^2t^2+r^2}}=\frac{1}{r}\int_0^{\infty}e^{-t}\Sigma(-1)^n\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\left(\frac{h}{r}\right)^{2n}t^{2n}\,dt$$

$$=\frac{1}{r}\Sigma(-1)^n\{1.3\ldots(2n-1)\}^2\left(\frac{h}{r}\right)^{2n} \qquad 47$$

by which the value of the integral may be calculated if r be not too small. Hence if r be very great

$$V=\frac{S}{2\pi k}\frac{h^2}{r^3},$$

or the potential at a great distance from the electrode varies inversely as the cube of the distance.

We may solve similarly the problem in which the conducting mass is bounded by two parallel infinite planes, the metal plate $z=0$, and the plane $z=a$, and the source is a disk electrode of radius r, with its centre on the axis of z applied to the latter. As before a feebly conducting film is supposed to exist between the metal plate and the conducting substance.

We simply add a quantity w, as before, to the distribution of potential which could have existed if there had been no film. Thus by the solution for the infinite stratum with disk electrode worked out above we have

$$V=\frac{S}{2\pi k r_1}\int_0^\infty \frac{\sinh \lambda z}{\cosh \lambda a}\sin(\lambda r_1) J_0(\lambda r)\frac{d\lambda}{\lambda}+w. \qquad 48$$

The potential w must fulfil the conditions

$$\frac{\partial w}{\partial z}=0, \text{ for } z=a$$

(since the flow from the source-electrode is supposed unaffected by w)

$$h\frac{\partial V}{\partial z}-w=0, \text{ for } z=0,$$

besides of course the differential equation for points within the medium.

The first condition is satisfied if we take

$$w=\int_0^\infty 2\cosh\lambda(z-a)\,\phi(\lambda) J_0(\lambda r)\,d\lambda.$$

Also when $z=0$,

$$\begin{aligned} h\frac{\partial V}{\partial z}-w &= \frac{Sh}{2\pi k r_1}\int_0^\infty \frac{\sin(\lambda r_1)}{\cosh(\lambda a)} J_0(\lambda r)\,d\lambda \\ &\quad -2h\int_0^\infty \sinh(\lambda a)\,\lambda\phi(\lambda) J_0(\lambda r)\,d\lambda \\ &\quad -2\int \cosh(\lambda a)\,\phi(\lambda) J_0(\lambda r)\,d\lambda \\ &=0. \end{aligned}$$

Hence $$\phi(\lambda)=\frac{Sh}{4\pi k r_1}\frac{\sin(\lambda r_1)}{\cosh(\lambda a)\{\cosh(\lambda a)+h\lambda\sinh(\lambda a)\}},$$

and $$w=\frac{Sh}{2\pi k r_1}\int_0^\infty \frac{\cosh\lambda(z-a)\sin(\lambda r_1) J_0(\lambda r)\,d\lambda}{\cosh(\lambda a)\{\cosh(\lambda a)+h\lambda\sinh(\lambda a)\}}, \qquad 49$$

so that

$$V=\frac{S}{2\pi k r_1}\int_0^\infty \frac{\sinh(\lambda z)+h\lambda\cosh(\lambda z)}{\cosh(\lambda a)+h\lambda\sinh(\lambda a)}\sin(\lambda r_1)\,J_0(\lambda r)\frac{d\lambda}{\lambda}. \qquad 50$$

If we denote by V_a the potential at the disk electrode we have

$$V_a=\frac{S}{2\pi k r_1}\int_0^\infty \frac{\sinh(\lambda a)+h\lambda\cosh(\lambda a)}{\cosh(\lambda a)+h\lambda\sinh(\lambda a)}\sin(\lambda r_1)\,J_0(\lambda r)\frac{d\lambda}{\lambda},$$

and if the area of the electrode be very small

$$V_a=\frac{S}{2\pi k}\int_0^\infty \frac{\sinh(\lambda a)+h\lambda\cosh(\lambda a)}{\cosh(\lambda a)+h\lambda\sinh(\lambda a)}J_0(\lambda r)\,d\lambda. \qquad 51$$

This is the difference of potential between the electrodes, that is, the disk and the metal plate. Comparing it with the difference of potential for the same flow through the stratum of the conductor without the plate, that is with half the total difference of potential given by 23 for the two electrodes at distance $2a$, which is

$$\frac{S}{2\pi k}\int_0^\infty \tanh(\lambda a)\,J_0(\lambda r)\,d\lambda,$$

we see that it exceeds the latter by

$$\frac{Sh}{2\pi k}\int_0^\infty \frac{\lambda J_0(\lambda r)\,d\lambda}{\cosh(\lambda a)\{\cosh(\lambda a)+h\lambda\sinh(\lambda a)\}}.$$

The resistance of the compound stratum now considered is therefore

$$R=\frac{1}{4kr_1}-\frac{1}{2}\frac{\log 2}{\pi k a}+\frac{h}{2\pi k}\int_0^\infty \frac{\lambda J_0(\lambda r)\,d\lambda}{\cosh(\lambda a)\{\cosh(\lambda a)+h\lambda\sinh(\lambda a)\}}. \qquad 52$$

Since the resistance is between the plate and the electrode, which is taken as of very small radius, $J_0(\lambda r)=J_0(0)$ nearly, and so we put unity for $J_0(\lambda r)$ in the expansion just found. The last term is the resistance of the film between the plate and the conductor, and in the case of a liquid in a voltaic cell, kept from complete contact with the plate by the disengagement of gas, is the apparent resistance of polarization. Its approximate value, if a/h is capable of being taken as infinitely small, is

$$\frac{1}{2\pi k a}\log\frac{h}{a}.$$

The value of V in 50 can be expanded in a trigonometrical series so as to enable comparisons of the value of V to be made

for different values of r. A suitable form to assume is

$$V = \Sigma b_\mu \cos \mu \frac{z-a}{a}. \tag{53}$$

From this by the equation

$$h\frac{\partial V}{\partial z} - w = 0$$

which holds for $z = 0$, we obtain the condition

$$\cot \mu = \mu \frac{h}{a}. \tag{54}$$

This transcendental equation has an infinite number of positive roots which are the values of μ.

To find the expansion we put

$$\frac{\sinh(\lambda z) + h\lambda \cosh(\lambda z)}{\cosh(\lambda a) + h\lambda \sinh(\lambda a)} = \Sigma c_\mu \cos \mu \frac{z-a}{a}.$$

Multiplying both sides by $\cos\{\mu(z-a)/a\}$ and integrating from 0 to a we obtain from the left-hand side

$$\frac{a^2\lambda}{\mu^2 + \lambda^2 a^2},$$

and from the right

$$\tfrac{1}{2}c_\mu a\left(1 + \frac{\sin 2\mu}{2\mu}\right) = \tfrac{1}{2}c_\mu a\left(1 + \frac{ha}{a^2 + h^2\mu^2}\right)$$

by the relation 54. Thus

$$c_\mu = \frac{2a\lambda}{\mu^2 + \lambda^2 a^2}\,\frac{a^2 + h^2\mu^2}{a^2 + ha + h^2\mu^2}.$$

Hence

$$b_\mu = \frac{S}{2\pi k}\int_0^\infty c_\mu J_0(\lambda r)\,d\lambda = \frac{Sa(a^2 + h^2\mu^2)}{\pi k(a^2 + ha + h^2\mu^2)}\int_0^\infty \frac{J_0(\lambda r)\,\lambda d\lambda}{\mu^2 + \lambda^2 a^2}.$$

The expansion is thus

$$V = \frac{S}{\pi k a}\Sigma\frac{a^2 + h^2\mu^2}{a^2 + ha + h^2\mu^2}\cos \mu\frac{z-a}{a}\int_1^\infty \frac{e^{-\mu\frac{r}{a}\lambda}\,d\lambda}{\sqrt{\lambda^2 - 1}} \tag{55}$$

by 30 above.

The first root of 54 is smaller the greater h is, the second root is always greater than π. Thus if r be fairly great the first term of the series just written down will suffice for V. Hence for $z = a$ V has a considerable value at a distance from the axis.

The solution can be modified by a like process to that used above to suit the case of a cylinder of finite radius c. We have to add to V in this case a function V' which fulfils the conditions

$$\frac{\partial V'}{\partial z} = 0, \text{ for } z = a, \quad h\frac{\partial V'}{\partial z} = V', \text{ for } z = 0,$$

$$\frac{\partial V}{\partial r} + \frac{\partial V'}{\partial r} = 0, \text{ for } r = c,$$

and satisfies the general differential equation. The reader may verify that if in the quantities $L_1(r), \ldots, L_1(c), \ldots$, of 32 above, κ be replaced by μ/a,

$$V + V'$$
$$= \frac{S}{\pi k a} \Sigma \frac{a^2 + h^2\mu^2}{a^2 + ha + h^2\mu^2} \frac{M_2(c) L_1(r) - M_1(c) L_2(r)}{M_2(c)} \cos\mu \frac{z-a}{a}. \qquad 56$$

As a final and very instructive example of the use of Fourier-Bessel expansions we take the problem of the flow of electricity in a right cylindrical conductor when the electrodes are placed on the same generating line of the cylindrical surface, at equal distances from the middle cross-section of the cylinder. We shall merely sketch the solution, leaving the reader to fill in the details of calculation.

The differential equation to be satisfied by the potential in this case is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0. \qquad 57$$

If the electrodes be supposed to be small equal rectangular disks, having their sides parallel to generating lines and ends of the cylinder, and the radius be unity, the surface conditions to be satisfied are summed up in the equations

$$\frac{\partial V}{\partial z} = 0, \text{ for } z = \pm a, \quad \frac{\partial V}{\partial r} = \Phi,$$

where

$$\Phi = \pm c \text{ for } \begin{cases} -\phi < \theta < +\phi \\ +\beta < z < \beta + \delta \\ -\beta > z > -(\beta + \delta) \end{cases}$$

$$\Phi = 0, \text{ for all other points.}$$

The distances of the centres of the electrodes from the central cross-section are here $\pm(\beta + \frac{1}{2}\delta)$ and the angle subtended at the axis by their breadth is 2ϕ, while the height of the cylinder is $2a$.

We have first to find an expression for Φ which fulfils these conditions. This can be obtained by Fourier's method and the result is

$$\Phi = \frac{4c}{\pi^2}\left\{\phi + 2\sum_1^\infty \frac{1}{n}\sin n\phi \cos n\theta\right\}\sum_0^\infty \frac{1}{2m+1}$$

$$\left\{\cos\frac{(2m+1)\pi}{2a}\beta - \cos\frac{(2m+1)\pi}{2a}(\beta+\delta)\right\}\sin\frac{(2m+1)\pi}{2a}z.$$

Now assume

$$V = \sum_m\sum_n A_{m,n}\psi(r)\sin\frac{(2m+1)\pi z}{2a}\cos n\theta, \qquad 58$$

and the differential equation 57 will be satisfied if $\psi(r)$ be a function of r which satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \left\{\frac{n^2}{r^2} + \left(\frac{2m+1}{2a}\pi\right)^2\right\}u = 0.$$

Hence we put

$$\psi(r) = J_n\left(i\frac{2m+1}{2a}\pi r\right) = J_n(ix). \qquad 59$$

To complete the solution the constant $A_{m,n}$ must be chosen so as to ensure the fulfilment of the surface condition. This clearly is done by writing

$$A_{m,n} = \frac{4c}{\pi^2}\frac{1}{2m+1}\frac{2\sin n\phi}{n\psi'(1)}$$

$$\left\{\cos\frac{(2m+1)\pi}{2a}\beta - \cos\frac{(2m+1)\pi}{2a}(\beta+\delta)\right\}$$

in which when $n=0$, ϕ is to be put instead of $2\sin n\phi/n$.

To find the effect of making the electrodes very small we substitute

$$2\sin\frac{(2m+1)\pi}{2a}(\beta+\tfrac{1}{2}\delta)\sin\frac{(2m+1)\pi}{4a}\delta$$

for the cosines in the value $A_{m,n}$, and $\phi\delta$ for

$$\frac{\sin n\phi}{n}\frac{\sin\{(2m+1)\pi\delta/4a\}}{(2m+1)\pi/4a}.$$

Remembering that $c\phi\delta$ is finite, and therefore putting

$$4c\phi\delta/\pi^2 = 1,$$

we get the solution

$$V = \frac{\pi}{2a} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \frac{\psi(r)}{\psi'(1)} \cos n\theta \sin \frac{(2m+1)\pi\beta}{2a} \sin \frac{(2m+1)\pi z}{2a} \qquad 60$$

where $\epsilon_0 = 1,\ \epsilon_1 = \epsilon_2 = \epsilon_3 = \ldots = \epsilon_n = \ldots = 2.$

For an infinitely long cylinder we can obtain the solution by putting in 60

$$\frac{\pi}{a} = d\lambda, \quad \frac{(2m+1)\pi}{2a} = \lambda,$$

and replacing summation by integration. Thus we obtain

$$V = \tfrac{1}{2} \sum_{0}^{\infty} \epsilon_n \cos n\theta \int_0^{\infty} \frac{J_n(i\lambda r)}{i\lambda J_n'(i\lambda)} \sin(\lambda\beta) \sin(\lambda z)\, d\lambda, \qquad 61$$

ϵ_0 as before being 1, and all the others 2.

The reader may verify as another example that if the electrodes be applied at the central cross-section at points for which $\theta = \pm\alpha$, the potential is given by

$$V = \sum_{1}^{\infty} \frac{r^n}{n} \sin n\alpha \sin n\theta + 2 \sum_{1}^{\infty} \sum_{1}^{\infty} \frac{J_n\left(i\dfrac{m\pi}{a} r\right)}{iJ_n'\left(i\dfrac{m\pi}{a}\right)} \frac{1}{m} \sin n\alpha \sin n\theta \cos \frac{m\pi z}{a}.$$

If a be infinitely small the second part of this expression vanishes and the first term can be written

$$V = \tfrac{1}{4} \log \frac{1 - 2r\cos(\alpha+\theta) + r^2}{1 - 2r\cos(\alpha-\theta) + r^2},$$

which agrees with an expression given by Kirchhoff (*Pogg. Ann.* Bd. 64, 1845) for the potential at any part of a circular disk with a source and a sink in its circumference.

The reader may refer to another paper by Weber (*Crelle*, Bd. 76, 1873) for the solution of some more complicated problems of electric flow, for example a conducting cylinder covered with a coaxial shell of relatively badly conducting fluid, the two electrodes being in the fluid and core respectively; and a cylindrical core covered with a coaxial cylinder of material of conductivity comparable with that of the core.

Each of Weber's papers contains a very valuable introductory analysis dealing for the most part with definite integrals involving Bessel Functions. Several of his results are included in the Examples at the end of this volume.

CHAPTER XIII.

PROPAGATION OF ELECTROMAGNETIC WAVES ALONG WIRES.

THE equations of the electromagnetic field were first given by Maxwell in 1865*. They have since been used in a somewhat modified form with great effect by Hertz and by Heaviside in their researches on the propagation of electromagnetic waves. The modification used by these writers is important as showing the reciprocal relation which exists between the electric and the magnetic force, and enables the auxiliary function called the vector-potential to be dispensed with in most investigations of this nature.

If $P, Q, R, \alpha, \beta, \gamma$ denote the components of electric and magnetic forces in a medium of conductivity k, electric inductive capacity κ, and magnetic inductive capacity μ, the equations referred to are

$$\left.\begin{aligned}\left(k+\frac{\kappa}{4\pi}\frac{\partial}{\partial t}\right)P&=\frac{1}{4\pi}\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}\right)\\ \left(k+\frac{\kappa}{4\pi}\frac{\partial}{\partial t}\right)Q&=\frac{1}{4\pi}\left(\frac{\partial\alpha}{\partial z}-\frac{\partial\gamma}{\partial x}\right)\\ \left(k+\frac{\kappa}{4\pi}\frac{\partial}{\partial t}\right)R&=\frac{1}{4\pi}\left(\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)\end{aligned}\right\} \qquad 1$$

and

$$\left.\begin{aligned}\frac{\mu}{4\pi}\frac{\partial\alpha}{\partial t}&=-\frac{1}{4\pi}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\\ \frac{\mu}{4\pi}\frac{\partial\beta}{\partial t}&=-\frac{1}{4\pi}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\\ \frac{\mu}{4\pi}\frac{\partial\gamma}{\partial t}&=-\frac{1}{4\pi}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\end{aligned}\right\}. \qquad 2$$

* On the Electromagnetic Field, *Phil. Trans.* 1865, *Electricity and Magnetism*, Vol. II. Chap. XX.

From these may be derived the equations

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0, \qquad 3$$

$$\frac{\partial \alpha}{\partial x}+\frac{\partial \beta}{\partial y}+\frac{\partial \gamma}{\partial z}=0. \qquad 4$$

The first of these expresses that there is no electrification at the point x, y, z, and the second that the magnetic force, being purely inductive, fulfils the solenoidal condition at every point, except of course at the origin of the disturbance.

At the surface of separation between two media the normal components of the magnetic induction, and the tangential components of the magnetic force, are continuous. The tangential components of electric force are also continuous.

From the equations given above the equations of propagation of an electromagnetic wave can be at once derived. Eliminating Q and R by means of the first of (2), the second and the third of (1) and (4), we get

$$4\pi\mu k\frac{\partial \alpha}{\partial t}+\kappa\mu\frac{\partial^2 \alpha}{\partial t^2}=\nabla^2\alpha, \qquad 5$$

and similarly two equations of the same form for β and γ. These are the equations of propagation of magnetic force.

By a like process we obtain the equations of propagation of electric force

$$\left.\begin{array}{l} 4\pi\mu k\dfrac{\partial P}{\partial t}+\kappa\mu\dfrac{\partial^2 P}{\partial t^2}=\nabla^2 P \\ \quad\quad\;\; \&c. \qquad\quad \&c. \end{array}\right\} \qquad 6$$

Now for the case of propagation with a straight wire as guide in an isotropic medium, the field is symmetrical round the wire at every instant. Therefore there is no component of electric force at right angles to a plane coinciding with the axis. From this it follows by the equations connecting the forces, that the magnetic force at any point in a plane coinciding with the axis is at right angles to that plane. The lines of magnetic force are therefore circles round the wire as axis.

Thus we may choose the axis of x as the axis of symmetry, and consider only two components of electric force, one P, parallel to the axis, and another R, from the axis in a plane passing through it. We shall denote the distance of the point considered from the

origin along the axis by x, and its distance from the axis by ρ, and shall use for the magnetic force at the same point the symbol H, which will thus correspond to the β of equations 1 and 2.

From 1 and 2 we get for our special case the equations

$$4\pi kP + \kappa\frac{\partial P}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H), \tag{7}$$

$$4\pi kR + \kappa\frac{\partial R}{\partial t} = \frac{\partial H}{\partial x}, \tag{8}$$

$$\mu\frac{\partial H}{\partial t} = \frac{\partial R}{\partial x} - \frac{\partial P}{\partial\rho}. \tag{9}$$

Eliminating first H and R from these equations we find for the differential equation satisfied by P

$$4\pi\mu k\frac{\partial P}{\partial t} + \kappa\mu\frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial P}{\partial\rho}. \tag{10}$$

Eliminating H and P we see that R must be taken so as to satisfy a slightly different equation, namely,

$$4\pi\mu k\frac{\partial R}{\partial t} + \kappa\mu\frac{\partial^2 R}{\partial t^2} = \frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho} - \frac{1}{\rho^2}R. \tag{11}$$

Finally, we easily find in the same way that H satisfies a differential equation precisely the same as 11.

In dealing with the problem we shall suppose at first that the wire has a certain finite radius, and is surrounded at a distance by a coaxial conducting tube which may be supposed to extend to infinity in the radial direction. There will therefore be three regions of the field to be considered, the wire, the outside conducting tube, and the space between them. The differential equations found above are perfectly general and apply, with proper values of the quantities k, μ, κ, to each region.

Taking first the space between the two conductors we shall suppose it filled with a perfectly insulating isotropic substance. The appropriate differential equations are therefore obtained by putting $k = 0$, in 10 and 11.

If the electric and magnetic forces be simply periodic with respect to x and t, each will be of the form

$$f(\rho)\,e^{(mx-nt)i}.$$

Let
$$P = ue^{(mx-nt)i},$$
$$R = ve^{(mx-nt)i},$$

where u, v denote the values of $f(\rho)$ for these two quantities. Substituting in 10, remembering that $k=0$, we find

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} - (m^2 - \kappa\mu n^2)\, u = 0. \qquad 12$$

The quantity $m^2 - \kappa\mu n^2$ is in general complex since mi includes a real factor which gives the alteration of amplitude with distance travelled by the wave along the wire. On the other hand n is essentially real being 2π times the frequency of the vibration.

If the wave were not controlled by the wire we should have in the dielectric $m^2 - \kappa\mu n^2 = 0$. The velocity of propagation of an electromagnetic disturbance in a medium of capacities κ, μ is according to theory $\sqrt{1/\kappa\mu}$; and this velocity has, for air at least, been proved to be that of light.

If we denote $m^2 - \kappa\mu n^2$ by p^2 and write ξ for $p\rho i$, 12 becomes

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial u}{\partial \xi} + u = 0, \qquad 13$$

which is the differential equation of the Bessel function of zero order $J_0(\xi)$.

In precisely the same way we get from 11 the equation

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial v}{\partial \xi} + \left(1 - \frac{1}{\xi^2}\right) v = 0, \qquad 14$$

the differential equation of the Bessel function of order 1, namely $J_1(\xi)$.

An equation of the same form as 14 is obtained in a similar manner for H.

Turning now to the conductors we suppose that in them κ is small in comparison with k. In ordinary conductors κ/k is about 10^{-17} in order of magnitude, so that we may neglect the displacement currents represented by the second terms on the left in equations 1. We thus obtain the proper differential equations by substituting $m^2 - 4\pi k\mu n i$ for p^2. We shall denote this by q^2 and write the equations

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta} + u = 0, \tag{15}$$

$$\frac{\partial^2 v}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial v}{\partial \eta} + \left(1 - \frac{1}{\eta^2}\right) v = 0, \tag{16}$$

where $$\eta = q\rho i = \rho i\sqrt{m^2 - 4\pi k\mu n i}.$$

Two values of q and η will be required, one for the wire and the other for the outer conductor; we shall denote these by q_1, η_1, q_2, η_2, respectively.

The general solution of 13 is as we have seen above, p. 11,

$$u = aJ_0(\xi) + bY_0(\xi), \tag{17}$$

where a and b are arbitrary constants to be determined to suit the conditions of the problem. The solution of 15 has of course the same form, with η substituted for ξ.

For very large values of η both $J_0(\eta)$ and $Y_0(\eta)$ become infinite, and we have to choose the arbitrary constants so that u may vanish in the outer conductor when ρ, and therefore η, is very great. It is clear from the value of $Y_n(x)$ given at p. 40 above for large values of the argument, that this condition can only be fulfilled if

$$a = b(\gamma - \log 2),$$

where γ is Euler's constant, it being understood that the real part of the argument is positive.

Again, for small values of the argument Y_0 becomes very great, so that in the wire we must put $b = 0$, since η there vanishes.

We thus get for the value of P in the three regions, the wire, the dielectric, and the outer conductor, the equations

$$P = AJ_0(\eta_1)e^{(mx-nt)i}, \tag{18}$$

$$P = \{BJ_0(\xi) + CY_0(\xi)\}e^{(mx-nt)i}, \tag{19}$$

$$P = D\{(\gamma - \log 2)J_0(\eta_2) + Y_0(\eta_2)\}e^{(mx-nt)i}, \tag{20}$$

in which A, B, C, D are constants to be determined by means of the conditions which hold at the surfaces of separation between the adjacent regions.

From the value of P we can easily obtain the component R at right angles to the axis. Since all the quantities are periodic 8 and 9 may be written in the form

$$(4\pi k - \kappa ni)\,R = miH$$

$$-\,\mu niH = miR - \frac{\partial P}{\partial \rho}.$$

Eliminating first H, then R, between these equations we obtain

$$R = -\frac{mi}{m^2 - \mu\kappa n^2 - 4\pi\mu kni}\frac{\partial P}{\partial \rho}, \qquad 21$$

$$H = \frac{-(4\pi k - \kappa ni)}{m^2 - 4\pi\mu kni - \mu\kappa n^2}\frac{\partial P}{\partial \rho}. \qquad 22$$

Thus if in the dielectric we put $k = 0$, and write p^2 for $m^2 - \mu\kappa n^2$, we get from 19, remembering that $p\rho i = \xi$,

$$R = \frac{m}{p}\frac{\partial P}{\partial \xi} = \frac{m}{p}\{BJ_0'(\xi) + CY_0'(\xi)\}\,e^{(mx-nt)i}, \qquad 23$$

$$H = -\frac{\kappa n}{p}\frac{\partial P}{\partial \xi}$$

$$= -\frac{\kappa n}{p}\{BJ_0'(\xi) + CY_0'(\xi)\}\,e^{(mx-nt)i}. \qquad 24$$

In the wire on the other hand where

$$q_1\rho i = \eta_1, \ [q^2 = m^2 - 4\pi\mu kni],$$

we have

$$R = \frac{m}{q_1}\frac{\partial P}{\partial \eta_1} = \frac{m}{q_1}AJ_0'(\eta_1)\,e^{(mx-nt)i}, \qquad 25$$

$$H = -\frac{4\pi k_1 i}{q_1}\frac{\partial P}{\partial \eta_1} = -\frac{4\pi k_1 i}{q_1}AJ_0'(\eta_1)e^{(mx-nt)i}. \qquad 26$$

Lastly in the outer conductor we have, writing, as at p. 91, $G_0(\eta_2)$ for

$$-\{(\gamma - \log 2)\,J_0(\eta_2) + Y_0(\eta_2)\},$$

$$R = \frac{m}{q_2}\frac{\partial P}{\partial \eta_2} = -\frac{m}{q_2}DG_0'(\eta_2)\,e^{(mx-nt)i}, \qquad 27$$

$$H = -\frac{4\pi k_2 i}{q_2}\frac{\partial P}{\partial \eta_2} = \frac{4\pi k_2 i}{q_2}DG_0'(\eta_2)\,e^{(mx-nt)i}. \qquad 28$$

We now introduce the boundary conditions, namely that the tangential electric force and the tangential magnetic force are continuous. From the latter condition it follows that the lines of magnetic force, being circles round the axis of the wire in the dielectric, are so also in the wire and also in the outer conductor.

These conditions expressed for the surface of the wire give for $\rho = a_1$

$$\left.\begin{aligned} BJ_0(\xi) + CY_0(\xi) &= AJ_0(\eta_1), \\ \frac{\kappa n}{p}\left\{BJ_0'(\xi) + CY_0'(\xi)\right\} &= \frac{4\pi k_1 i}{q_1} AJ_0'(\eta_1), \end{aligned}\right\} \qquad 29$$

and for $\rho = a_2$

$$\left.\begin{aligned} BJ_0(\xi) + CY_0(\xi) &= -DG_0(\eta_2), \\ \frac{\kappa n}{p}\left\{BJ_0'(\xi) + CY_0'(\xi)\right\} &= -\frac{4\pi k_2 i}{q_2} DG_0'(\eta). \end{aligned}\right\} \qquad 30$$

Denoting the values of $J_0(\xi)$, $Y_0(\xi)$ for $\rho = a_1$, and $\rho = a_2$ by $J_0(\xi)_1$, $Y_0(\xi)_1$, $J_0(\xi)_2$, $Y_0(\xi)_2$, and eliminating the four constants A, B, C, D by means of the equations just written down, we find

$$\frac{4\pi k_1 piJ_0'(\eta_1)\,J_0(\xi)_1 - \kappa n q_1 J_0(\eta_1)\,J_0'(\xi)_1}{4\pi k_2 piG_0'(\eta_2)\,J_0(\xi)_2 - \kappa n q_2 G_0(\eta_2)\,J_0'(\xi)_2}$$
$$= \frac{4\pi k_1 piJ_0'(\eta_1)\,Y_0(\xi)_1 - \kappa n q_1 J_0(\eta_1)\,Y_0'(\xi)_1}{4\pi k_2 piG_0'(\eta_2)\,Y_0(\xi)_2 - \kappa n q_2 G_0(\eta_2)\,Y_0'(\xi)_2}. \qquad 31$$

Considering first long waves of low frequency and remembering that $\kappa\mu$ is $1/V^2$ where V is the velocity of light in the dielectric, we see that p reduces to m nearly, and the real part of m^2 is $4\pi^2/\lambda^2$ where λ is the wave-length. Thus if a_1 is not large pa_1 is very small. Also if a_2, the radius of the insulating cylinder, is moderately small, pa_2 is also small.

Now when a_1, a_2, are small the approximate values of the functions at the cylindrical boundaries are

$$J_0(\xi) = 1,\ J_0(\eta) = 1,\ Y_0(\xi) = \log\xi,\ G_0(\eta) = -\log\frac{e^{\gamma}\eta}{2},$$

$$J_0'(\xi) = -\tfrac{1}{2}\xi,\ J_0'(\eta) = -\tfrac{1}{2}\eta,\ Y_0'(\xi) = \frac{1}{\xi},\ G_0'(\eta) = -\frac{1}{\eta}.$$

Using these values for the J and Y functions in equation 31, putting for brevity ϕ, χ, for the ratios $J_0(\eta_1)/J_0'(\eta_1)$, $G_0(\eta_2)/G_0'(\eta_2)$, and α_1, α_2 for $4\pi k_1$, $4\pi k_2$, and neglecting terms involving the factors $a_1^2 a_2$, $a_1 a_2^2$ in comparison with others involving the factor $a_1 a_2$, we find after a little reduction

$$p^2 = \frac{\kappa n}{\log\dfrac{a_2}{a_1}}\left\{\frac{q_1}{\alpha_1 a_1}\phi - \frac{q_2}{\alpha_2 a_2}\chi + \tfrac{1}{2}\frac{\kappa n q_1 q_2}{\alpha_1\alpha_2 a_1 a_2}(a_2^2 - a_1^2)\,\phi\chi\right\}. \qquad 32$$

In all cases which occur in practice it may be assumed that $|q^2|$ is approximately $4\pi\mu k n$. Further $\kappa n = n/\mu V^2$, so that the last term within the brackets in the preceding expression is

$$-\frac{i}{8\pi}\frac{n^2}{\mu V^2}\frac{\sqrt{\mu_1\mu_2}}{\sqrt{k_1 k_2}}\frac{a_2{}^2 - a_1{}^2}{a_1 a_2}\phi\chi.$$

The first of the other two terms within the brackets is

$$\frac{\sqrt{-4\pi\mu_1 k_1 n i}}{4\pi k_1 a_1}\phi = \frac{1-i}{\sqrt{2}}\frac{\sqrt{\mu_1 n}}{\sqrt{4\pi k_1}}\frac{\phi}{a_1},$$

and for small values of a_1, a_2, the values of ϕ, χ, are large. Hence the modulus of the first term, unless the frequency, $n/2\pi$, of the vibrations is very great, is large in comparison with that of the third term. The same thing can be proved of the second term and the third. Hence the third term, on the supposition of low frequency and small values of a_1, a_2, may be neglected in comparison with the first and second. Equation 32 thus reduces to

$$p^2 = \frac{n^{\frac{3}{2}}}{\mu V^2}\frac{1-i}{\sqrt{8\pi}}\left(\frac{\sqrt{\mu_1}}{\sqrt{k_1}}\frac{\phi}{a_1} - \frac{\sqrt{\mu_2}}{\sqrt{k_2}}\frac{\chi}{a_2}\right)\frac{1}{\log\frac{a_2}{a_1}}. \qquad 33$$

Let now the frequency be so small that $q_1 a_1$ is very small. Then we have

$$\phi = \frac{J_0(\eta_1)}{J_0'(\eta_1)} = -\frac{2}{q_1 a_1 i},$$

and

$$\chi = \frac{G_0(\eta_2)}{G_0'(\eta_2)} = q_2 a_2 i \log\frac{e^{\gamma} q_2 a_2 i}{2},$$

and it is clear that the second term of 33 bears to the first only a very small ratio unless a_2 be very great indeed. In this case then we may neglect the second term in comparison with the first, and we get

$$p^2 = \frac{ni}{\mu V^2}\frac{1}{2\pi a_1{}^2 k_1}\frac{1}{\log\frac{a_2}{a_1}}, \qquad 34$$

or, since $p^2 = m^2 - \kappa\mu n^2 = m^2 - n^2/V^2$,

$$m^2 = \frac{n^2}{V^2}\left(1 + \frac{i}{2\pi\mu k_1 n a_1{}^2}\frac{1}{\log\frac{a_2}{a_1}}\right), \qquad 35$$

The modulus of the second term in the brackets is great in comparison with unity, and hence if we take only the imaginary part of m^2 as given by 35 we shall get a value of m, the real part of which is great in comparison with that which we should obtain if we used only the real part, that is we shall make the first approximation to m which 35 affords. Thus we write instead of 35

$$m^2 = \frac{n}{V^2}\frac{i}{2\pi\mu k_1 a_1^2}\frac{1}{\log\frac{a_2}{a_1}}. \qquad 36$$

But if r be the resistance of the wire, and c the capacity of the cable, each taken per unit of length,

$$r = 1/\pi a_1^2 k_1,\ c = \kappa/(2\log a_2/a_1)$$

(where κ is taken in electromagnetic units), and we have

$$m = \frac{1+i}{\sqrt{2}}\sqrt{nrc}, \qquad 36'$$

taking the positive sign.

This corresponds to a wave travelling with velocity $\sqrt{2n}/\sqrt{rc}$, and having its amplitude damped down to $1/e$ of its initial amount in travelling a distance $\sqrt{2}/\sqrt{nrc}$.

The other root of m^2 would give a wave travelling with the same speed but in the opposite direction, and with increasing amplitude. It is therefore left out of account.

We have thus fallen upon the ordinary case of slow signalling along a submarine or telephone cable, in which the electromagnetic induction may be neglected, and the result agrees with that found by a direct solution of this simple case of the general problem.

The velocity of phase propagation being proportional to the square root of the frequency of vibration, the higher notes of a piece of music would be transmitted faster than the lower, and the harmony might if the distance were great enough be disturbed from this cause. Further these higher notes are more rapidly damped out with distance travelled than the lower, and hence the relative strengths of the notes of the piece would be altered, the higher notes being weakened relatively to the lower.

We can now find the electric and magnetic forces. The electromotive intensity in the wire is given by

$$P = AJ_0(\eta_1)\, e^{(mx-nt)i},$$

where $\eta_1 = q_1\rho i$, the suffix denoting that ρ is less than the radius of the wire. But if the wire is, as we here suppose it to be, very thin, $J_0(\eta_1) = 1$, and the value of P is the same over any cross-section of the wire. Hence if γ_0 denote the total current in the wire at the plane $x = 0$, when $t = 0$, we have

$$AJ_0(\eta_1) = \gamma_0 r,$$

and therefore

$$P = \gamma_0 r e^{(mx-nt)i}. \qquad 37$$

Hence realizing we obtain

$$P = \gamma_0 r e^{-\sqrt{\frac{nrc}{2}}x} \cos\left(\sqrt{\frac{nrc}{2}}\, x - nt\right). \qquad 38$$

The radial electromotive intensity in the wire is given by 25, which is

$$R = \frac{m}{q_1} AJ_0'(\eta_1)\, e^{(mx-nt)i}.$$

But

$$\frac{m}{q_1} AJ_0'(\eta_1) = -\frac{m}{q_1}\gamma_0 r \tfrac{1}{2}\eta_1$$

$$= \gamma_0 \frac{1-i}{2\sqrt{2}} \rho r \sqrt{nrc},$$

so that

$$R = \frac{1-i}{2\sqrt{2}} \gamma_0 r\rho \sqrt{nrc}\, e^{(mx-nt)i}. \qquad 39$$

Again realizing we find

$$R = \tfrac{1}{2}\gamma_0 r\rho \sqrt{nrc}\, e^{-\sqrt{\frac{nrc}{2}}x} \cos\left(\sqrt{\frac{nrc}{2}}\, x - nt - \frac{\pi}{4}\right). \qquad 40$$

R therefore vanishes at the axis of the wire, and the electromotive intensity is there along the axis. Elsewhere R is sensible, and at the surface the ratio of its amplitude to that of P is $\frac{1}{2}a_1\sqrt{nrc}$.

The magnetic force in the wire is given by the equation [26 above]

$$H = -\frac{4\pi k_1 i}{q_1} AJ_0'(\eta_1)\, e^{(mx-nt)i},$$

which by what has gone before reduces to

$$H = -\frac{2}{a_1^2}\gamma_0 \rho e^{(mx-nt)i}. \qquad 41$$

The realized form of this is

$$H = -\frac{2}{a_1^2}\gamma_0\rho e^{-\sqrt{\frac{nrc}{2}}x}\cos\left(\sqrt{\frac{nrc}{2}}x - nt\right). \qquad 42$$

By a well-known theorem we ought to have numerically

$$2\pi\rho H = 4\pi\frac{\rho^2}{a^2}\gamma,$$

where γ is the total current at any cross-section. For γ we have here the equation

$$\gamma = \gamma_0 e^{(mx-nt)i}, \qquad 43$$

which when compared with 42 obviously fulfils the required relation numerically. Hence the results are so far verified.

We shall now calculate the forces in the dielectric. Putting $P_{0\rho}$ for the electromotive intensity at distance ρ from the axis in the plane $x=0$, and at time $t=0$, we find by the approximate values of the functions given at p. 148 above

$$B + C\log(p\rho i) = P_{0\rho}.$$

But at the surface of the wire $P_{0\rho} = \gamma_0 r$, where r denotes as before the resistance of the wire per unit length. Thus

$$B + C\log(pa_1 i) = \gamma_0 r.$$

Hence subtracting the former equation we find

$$P_{0\rho} = \gamma_0 r - C\log\frac{a_1}{\rho}.$$

C can be found from 29 by putting (by 18) $A = \gamma_0 r$ and eliminating B. Thus we obtain

$$C = 2\gamma_0\mu c r V^2$$

very approximately. Hence

$$P = \gamma_0 r\left(1 - 2\mu c V^2\log\frac{\rho}{a_1}\right)e^{(mx-nt)i}, \qquad 44$$

of which the realized form is

$$P = \gamma_0 r\left(1 - 2\mu c V^2\log\frac{\rho}{a_1}\right)e^{-\sqrt{\frac{nrc}{2}}x}\cos\left(\sqrt{\frac{nrc}{2}}x - nt\right). \qquad 45$$

From 23, 44, and 36', since $p = m$ nearly, we get

$$R = (1+i)\mu V^2 \gamma_0 \sqrt{\frac{2rc}{n}} \frac{1}{\rho} e^{(mx-nt)i}, \qquad 46$$

or retaining only the real part

$$R = 2\mu V^2 \gamma_0 \sqrt{\frac{rc}{n}} \frac{1}{\rho} e^{-\sqrt{\frac{nrc}{2}}x} \cos\left(\sqrt{\frac{nrc}{2}}x - nt + \frac{\pi}{4}\right). \qquad 47$$

Finally from 24 and 36 we have

$$H = -2\frac{\gamma_0}{\rho} e^{(mx-nt)}, \qquad 48$$

or

$$H = -2\frac{\gamma_0}{\rho} e^{-\sqrt{\frac{nrc}{2}}x} \cos\left(\sqrt{\frac{nrc}{2}}x - nt\right). \qquad 49$$

Thus the solution is completed for slow vibrations in a cable of small radius.

So far we have followed with certain modifications the analysis of Prof. J. J. Thomson, as set forth in his *Recent Researches in Electricity and Magnetism* (the Supplementary Volume to his Edition of Maxwell's *Electricity and Magnetism*). To that work the reader may refer for details of other applications to Electrical Oscillations. Reference should also be made to Mr Oliver Heaviside's important memoirs on the same subject, *Electrical Papers*, Vols. I. and II. *passim.*

We shall now obtain an expansion of $xJ_0(x)/J_0'(x)$ in ascending powers of x which will be of use in the discussion of the effective resistance and self-inductance in the case of a cable carrying rapidly alternating currents, and which is also useful in other applications when pa_1, q_1a_1, q_2a_2 are not very small.

Denoting the function $xJ_0(x)/J_0'(x)$ (or, for brevity, xJ_0/J_0') by u, we have by the relations proved at p. 13 above

$$u = x\frac{J_0}{J_0'} = -x\frac{J_0}{J_1}\left(= -2 + x\frac{J_2}{J_1}\right). \qquad 50$$

Hence

$$\begin{aligned}\frac{1}{u}\frac{du}{dx} &= \frac{1}{x} + \frac{J_0'}{J_0} - \frac{J_1'}{J_1} \\ &= \frac{1}{x} - \frac{J_1}{J_0} + \frac{1}{J_1}\left(\frac{1}{x}J_1 - J_0\right) \\ &= \frac{2}{x} - \frac{J_0}{J_1} - \frac{J_1}{J_0} \\ &= \frac{2}{x} + \frac{u}{x} + \frac{x}{u}.\end{aligned}$$

Therefore

$$x\frac{du}{dx}=u(u+2)+x^2. \qquad 51$$

Now guided by the value in brackets in 50 we assume

$$u=-2+\frac{x^2}{4}+a_4x^4+a_6x^6+\ldots$$

Then by 51

$$x\left(\frac{x}{2}+4a_4x^3+6a_6x^5+\ldots\right)=x^2+\left(\frac{x^2}{4}+a_4x^4+\ldots\right)\left(-2+\frac{x^2}{4}+\ldots\right).$$

Multiplying these expressions out and equating coefficients we find

$$\begin{aligned}4a_4&=-2a_4+\tfrac{1}{16},\\ 6a_6&=-2a_6+\tfrac{1}{2}a_4,\\ 8a_8&=-2a_8+\tfrac{1}{2}a_6+a_4^2,\\ 10a_{10}&=-2a_{10}+\tfrac{1}{2}a_8+2a_4a_6,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Hence

$$a_4=\frac{1}{2^5.3},\quad a_6=\frac{1}{2^9.3},\quad a_8=\frac{1}{2^9.3^2.5},\quad a_{10}=\frac{13}{2^{15}.3^3.5},\ldots.$$

Thus we have

$$x\frac{J_0(x)}{J_0'(x)}=-2+\frac{x^2}{4}+\frac{x^4}{2^5.3}+\frac{x^6}{2^9.3}+\frac{x^8}{2^9.3^2.5}+\frac{13x^{10}}{2^{15}.3^3.5}+\ldots \qquad 52$$

This expansion may be converted into a continued fraction, the successive convergents of which will give the value of the function to any desired degree of accuracy. The result, which may be verified by the reader, is

$$x\frac{J_0(x)}{J_0'(x)}=-2+\frac{x^2}{4-}\,\frac{x^2}{6-}\,\frac{x^2}{8-}\ldots. \qquad 53$$

Using 52 we obtain

$$iq_1a_1\frac{J_0(iq_1a_1)}{J_0'(iq_1a_1)}=-2-\frac{q_1^2a_1^2}{4}+\frac{q_1^4a_1^4}{2^5.3}-\frac{q_1^6a_1^6}{2^9.3}+\ldots.$$

Now approximately $q_1^2=-4\pi\mu_1k_1ni$ so that

$$iq_1a_1\frac{J_0(iq_1a_1)}{J_0'(iq_1a_1)}=-\left\{2+\frac{(4\pi\mu_1k_1na_1^2)^2}{2^5.3}-\frac{(4\pi\mu_1k_1na_1^2)^4}{2^9.3^2.5}+\ldots\right\}$$
$$+i\left\{\frac{4\pi\mu_1k_1na_1^2}{4}-\frac{(4\pi\mu_1k_1na_1^2)^3}{2^9.3}+\ldots\right\}. \qquad 54$$

The following table of values is given by Prof. J. J. Thomson:

$4\pi\mu_1 n a_1^2 k_1$	$iq_1 a_1 J_0(iq_1 a_1)/J_0'(iq_1 a_1)$
·5	$-2·002 + ·124i$
1	$-2·010 + ·250i$
1·5	$-2·024 + ·372i$
2	$-2·042 + ·50i$
2·5	$-2·064 + ·62i$
3	$-2·090 + ·74i$

This table shows that for values of $4\pi\mu_1 n a_1^2 k_1$ up to unity -2 may still be taken as an approximation to $\eta_1 J_0(\eta_1)/J_0'\eta_1$ as above, p. 148. Thus the first term on the right of 33 is the same as before.

The last term on the right of 33 in the case of bare overhead wires depends on a large value of a_2, and we have to find an approximate value of $G_0(\eta_2)/G_0'(\eta_2)$ for this case. It has been shewn at p. 90, that $G_n(ix) = i^n K_n(x)$; and this relation holds when x is complex. Further, the semiconvergent expansion 142, p. 68, is valid when the real part of x is positive.

In the present case, since $\eta_2 = iq_2 a_2$, we have

$$x = q_2 a_2 = a_2 \sqrt{4\pi k_2 \mu_2 n}\,\sqrt{-i},$$

which, taking the same sign for each square root, gives

$$x = (1-i)\, a_2 \sqrt{2\pi\mu_2 k_2 n},$$

of which the real part is positive. Now we have

$$iG_0'(\eta_2) = K_0'(x), \text{ and } K_0'(x) = K_1(x).$$

Thus we get

$$\chi = \frac{G_0(\eta_2)}{G_0'(\eta_2)} = i\,\frac{K_0(x)}{K_0'(x)}$$
$$= -i$$

by 142. Hence

$$\sqrt{\frac{\mu_2}{k_2}}\frac{\chi}{a_2} = -\sqrt{\frac{\mu_2}{k_2}}\frac{i}{a_2}.$$

This is small in comparison with the first term unless k_2 be very small. Supposing the latter not to be the case, we have the same solution as before.

[The approximation stated above for $G_0(\eta)/G_0'(\eta)$, and others which hold for large values of the argument of the functions, are important. By the same process as that already used we can

prove, from the relation $J_n(ix)=i^n I_n(x)$ (139, p. 66), and the semiconvergent expansion 143, p. 68, that approximately

$$\frac{J_0(\eta)}{J_0'(\eta)}=i,$$

when $\eta = ix$, and x is a complex quantity of which the real part is positive, and of which the modulus is large.

Again it is worth noticing that, for any value of x with real part positive, we may write by p. 90

$$G_n(ix) = i^n K_n(x)$$

$$= i^n \sqrt{\frac{\pi}{2x}} \cos n\pi \,.\, e^{-x} \qquad 55$$

by 142, p. 68, if the modulus of x be very large.

Similarly

$$G_n'(ix) = i^{n-1} K_n'(x) = i^{n-1}\left\{K_{n-1}(x) - \frac{n}{x} K_n(x)\right\}$$

$$= i^{n-1}\sqrt{\frac{\pi}{2x}} \cos(n-1)\pi \,.\, e^{-x} \qquad 55'$$

approximately, if x have a very large modulus. Thus we get

$$\frac{G_n(ix)}{G_n'(ix)} = -i \qquad 56$$

of which $G_0(\eta)/G_0'(\eta) = -i$, is a particular case.

In the same circumstances

$$J_n(ix) = i^n I_n(x)$$

$$= i^n \sqrt{\frac{1}{2\pi x}}\, e^x, \qquad 57$$

and further

$$J_n'(ix) = i^{n-1}\left\{I_{n-1}(x) - \frac{n}{x} I_n(x)\right\}$$

$$= i^{n-1}\sqrt{\frac{1}{2\pi x}}\, e^x \left\{1 - \frac{4n^2+3}{8x} + \dots\right\}$$

$$= i^{n-1}\sqrt{\frac{1}{2\pi x}}\, e^x \qquad 57'$$

when the modulus of x is very great. Thus we have the result

$$\frac{J_n(ix)}{J_n'(ix)} = i \qquad 58'$$

already stated above for $n = 0$.]

As a further illustration the reader may work out the case of oscillations so rapid that both q_1a_1 and q_2a_2 are very large. Here

$$\phi = \frac{J_0(\eta_1)}{J_0'(\eta_1)} = i, \quad \chi = \frac{G_0(\eta_2)}{G_0'(\eta_2)} = -i,$$

so that by 33

$$p^2 = \frac{n^2}{V^2}\left(\frac{\mu_1}{q_1a_1} + \frac{\mu_2}{q_2a_2}\right)\frac{1}{\log\frac{a_2}{a_1}}.$$

Thus

$$m^2 = p^2 + \frac{n^2}{V^2} = \frac{n^2}{V^2}\left\{1 + \left(\frac{\mu_1}{q_1a_1} + \frac{\mu_2}{q_2a_2}\right)\frac{1}{\log\frac{a_2}{a_1}}\right\},$$

and approximately

$$m = \frac{n}{V}\left\{1 + \frac{i}{\sqrt{32\pi n}}\left(\sqrt{\frac{\mu_1}{a_1^2k_1}} + \sqrt{\frac{\mu_2}{a_2^2k_2}}\right)\frac{1}{\log\frac{a_2}{a_1}}\right\}. \qquad 59$$

The velocity of propagation is thus V, and the distance travelled, while the amplitude is diminishing to the fraction $1/e$ of its original value, is the product of n by the reciprocal of the coefficient of i within the brackets. The damping in this case is slow, since the imaginary part of m is of much smaller modulus than the real part. Here if $a_2^2k_2$ be small compared with $a_1^2k_1$, as in the case of a cable surrounded by sea water, the outside conductor will mainly control the damping, and nothing will be gained by using copper in preference to an inferior metal.

We shall now calculate the current density at different distances from the axis in a wire carrying a simply periodic current, and the effective resistance and self-inductance of a given length l of the conductor. Everything is supposed symmetrical about the axis of the wire.

By 18 we have for the axially directed electromotive intensity at a point in the wire distant $\rho\,(=\eta/iq)$ from the axis

$$P = AJ_0(\eta_1)\,e^{(mx-nt)i}. \qquad 60$$

This multiplied by k_1, the conductivity of the wire, gives an expression for the current density parallel to the axis of the wire at distance ρ from the axis.

If the value of P at the surface of the wire be denoted by P_{a_1},

$$P_{a_1} = AJ_0(\eta_1)\,e^{(mx-nt)i}, \qquad 61$$

$(\rho = a_1)$.

The magnetic force at the surface is

$$H_{a_1} = -\frac{4\pi k_1 i}{q_1} A J_0'(\eta_1) e^{(mx-nt)i},$$

$(\rho = a_1)$. Therefore if Γ be the total current in the wire we have $4\pi\Gamma = -2\pi a_1 H_{a_1}$, and so

$$\Gamma = \frac{2\pi k_1 a_1 i}{q_1} A J_0'(\eta_1) e^{(mx-nt)i}. \qquad 62$$

The electromotive intensity P is the resultant parallel to the axis of the impressed and induced electromotive intensities. To solve the problem proposed we must separate the part impressed by subtracting from P the induced part. Now the impressed electromotive force is the same all over any cross-section of the wire at a given instant, and will therefore be determined if we find it for the surface. But since the induced electromotive intensity due to any part of the current is directly proportional to its time-rate of variation, the induced electromotive intensity at the surface must be directly proportional to the time-rate of variation of the whole current in the wire. Hence by 61 if E denote the impressed electromotive intensity

$$E - A'\dot{\Gamma} = P_{a_1},$$

where $A'\dot{\Gamma}$ (A' = a constant) is put for the induced intensity parallel to the axis at the surface. Thus

$$E = A\left\{J_0(\eta_1) - n\frac{2\pi k_1 a_1}{q_1} A' J_0'(\eta_1)\right\} e^{(mx-nt)i}$$

$$= -\left\{\frac{iq_1 a_1}{2\pi k_1 a_1^2}\frac{J_0(\eta_1)}{J_0'(\eta_1)} - niA'\right\}\Gamma. \qquad 63$$

Putting r for the resistance $(=1/\pi a_1^2 k_1)$ of unit length of the wire and using the expansion above, we get since $q_1^2 = -4\pi\mu_1 k_1 ni$,

$$E = r\left(1 + \frac{1}{12}\frac{\mu_1^2 n^2}{r^2} - \frac{1}{180}\frac{\mu_1^4 n^4}{r^4} + \dots\right)\Gamma$$

$$-in\left\{-A' + \mu_1\left(\frac{1}{2} - \frac{1}{48}\frac{\mu_1^2 n^2}{r^2} + \frac{13}{8640}\frac{\mu_1^4 n^4}{r^4} - \dots\right)\right\}\Gamma. \qquad 64$$

Or taking the impressed difference of potential V between the two ends of a length l of the wire the resistance of which is R we have

$$V = R\left(1 + \frac{1}{12}\frac{\mu_1^2 n^2 l^2}{R^2} - \frac{1}{180}\frac{\mu_1^4 n^4 l^4}{R^4} + \dots\right)\Gamma$$

$$-in\left\{-lA' + \mu_1\left(\frac{l}{2} - \frac{1}{48}\frac{\mu_1^2 n^2 l^3}{R^2} + \frac{13}{8640}\frac{\mu_1^4 n^4 l^5}{R^4} - \dots\right)\right\}\Gamma. \qquad 65$$

If we denote the series in brackets in the first and second terms respectively by R', L' we get

$$V = R'\Gamma + L'\dot{\Gamma}. \qquad 66$$

Thus R' and L' are the effective resistance and self-inductance of the length l of the wire.

It remains to determine the constant A'. If there be no displacement current in the dielectric comparable with that in the wire, a supposition sufficiently nearly in accordance with the fact for all practical purposes, and the return current be capable of being regarded as in a highly conducting skin on the outside of the dielectric, so that there is no magnetic force outside, we can find A' in the following manner. The inductive electromotive force per unit length in the conductor at any point is then equal to the rate of variation of the surface integral of magnetic force taken per unit length in the dielectric at that place. Now, if there is no displacement current, H will be in circles round the axis of the wire, and will be inversely as the radius of the circle at any point, since

$$2\pi r H = -4\pi\Gamma.$$

Thus if H_r be the magnetic force at distance r from the axis of the wire

$$\dot{H}_r = -\frac{4\pi k_1 n}{q_1} A J_0'(\eta_1)\frac{a_1}{r}\, e^{(mx-nt)\,i},$$

and $$\int_{a_1}^{a_2} \dot{H}_r dr = -\frac{4\pi k_1 n a_1}{q_1} A J_0'(\eta_1) \log\frac{a_2}{a_1}\, e^{(mx-nt)\,i}. \qquad 67$$

But this last expression by what has been stated above is $A'\dot{\Gamma}$, and $\dot{\Gamma}$ is given by 62.

Thus we obtain

$$A' = -2\log\frac{a_2}{a_1}$$

and $$L' = 2l\log\frac{a_2}{a_1} + l\mu_1\left(\frac{1}{2} - \frac{1}{48}\frac{\mu_1^2 n^2 l^2}{R^2} + \frac{13}{8640}\frac{\mu_1^4 n^4 l^4}{R^4} - \dots\right). \qquad 68$$

If $J_0(x\sqrt{i})$ be denoted by $X - Yi$, $\partial X/\partial x$, $\partial Y/\partial y$ by X', Y', and $x = 2\sqrt{\mu_1 n/r}$, then by 63 we easily find

$$R' = \frac{x}{2}\frac{XY' - X'Y}{X'^2 + Y'^2} R, \qquad 69$$

$$L' = 2l \log \frac{a_2}{a_1} + \frac{xlr}{2n} \frac{XX' + YY'}{X'^2 + Y'^2}, \qquad 70$$

a form in which the values of R' and L' are easily calculated for any given values of x and n from the Table of $J_0(x\sqrt{i})$ given at the end of the book.

Equation 65 shows the effect of μ_1 on R' and L' at different frequencies. If however the frequency be very great, we must put in 63 $J_0(\eta_1)/J_0'(\eta_1) = i$. We find for this case

$$R' = \sqrt{\tfrac{1}{2}\mu_1 nlR}, \quad L' = \sqrt{\frac{\mu_1 Rl}{2n}} - lA' \ldots \qquad 71$$

Thus in the limit R' is indefinitely great, and L' reduces to the constant term $-lA'$. The current is now insensible except in an infinitely thin stratum at the surface of the wire.

The problem of electrical oscillations has been treated somewhat differently by Hertz in his various memoirs written in connection with his very remarkable experimental researches*. He discussed first the propagation, in an unlimited dielectric medium, of electric and magnetic disturbances from a vibrator consisting of two equal plates or balls connected by a straight wire with a spark-gap in the middle, and, secondly, the propagation in the same medium of disturbances generated by such a vibrator guided by a long straight wire. The action of the vibrator simply consisted in a flow of electricity alternately from one plate or ball to the other, set up by an initially impressed difference of potential between the two conductors.

Taking the simple case first as an introduction to the second, which we wish to give some account of here, we may take the vibrator as an electric doublet, that is as consisting electrically of two equal and opposite point-charges at an infinitesimal distance apart, and having the line joining them along the axis of x, and the origin midway between them. It is clear in this case that everything is symmetrical about the axis of x, that the electric forces lie in planes through the axis, and that the lines of magnetic force are circles round the wire.

The equations of motion are those given on p. 142 above. By

* See Hertz's *Untersuchungen über die Ausbreitung der elektrischen Kraft*, J. A. Barth, Leipzig, 1892; or *Electric Waves* (the English Translation of the same work, by Mr D. E. Jones), Macmillan and Co., London, 1893.

symmetry the component α of magnetic force in the medium is zero, and the equation

$$\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}=0$$

holds, connecting the other two components. This shows that $\beta dz-\gamma dy$ is a complete differential of some function of y, z. In Hertz's notation we take this function as $\partial\Pi/\partial t$, so that

$$\beta=-\frac{\partial^2\Pi}{\partial z\partial t},\quad \gamma=\frac{\partial^2\Pi}{\partial y\partial t}. \tag{72}$$

The equations of motion become then

$$\kappa\frac{\partial P}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial^2\Pi}{\partial y^2}+\frac{\partial^2\Pi}{\partial z^2}\right),$$

$$\kappa\frac{\partial Q}{\partial t}=-\frac{\partial^3\Pi}{\partial t\partial x\partial y},$$

$$\kappa\frac{\partial R}{\partial t}=-\frac{\partial^3\Pi}{\partial t\partial x\partial z},$$

which declare that the quantities

$$\kappa P-\frac{\partial^2\Pi}{\partial y^2}+\frac{\partial^2\Pi}{\partial z^2},\quad \kappa Q+\frac{\partial^2\Pi}{\partial x\partial y},\quad \kappa R+\frac{\partial^2\Pi}{\partial x\partial z},$$

are independent of t. The propagation of waves in the medium therefore will not be affected if we suppose each of these quantities to have the value zero. Thus we assume as the fundamental equations

$$\kappa P=\left(\frac{\partial^2\Pi}{\partial y^2}+\frac{\partial^2\Pi}{\partial z^2}\right),$$

$$\kappa Q=-\frac{\partial^2\Pi}{\partial x\partial y},$$

$$\kappa R=-\frac{\partial^2\Pi}{\partial x\partial z}.$$

Using these in the equations of magnetic force, 2, we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial^2\Pi}{\partial t^2}-\frac{1}{\kappa\mu}\nabla^2\Pi\right)=0,$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial t^2}-\frac{1}{\kappa\mu}\nabla^2\Pi\right)=0,$$

which show that the quantity in brackets is a function of x and t

only. Thus we write

$$\frac{\partial^2 \Pi}{\partial t^2} - \frac{1}{\kappa\mu} \nabla^2 \Pi = f(x, t).$$

It is easy to see that we may put $f(x, t) = 0$ without affecting the electric and magnetic fields, and the equation of propagation is

$$\frac{\partial^2 \Pi}{\partial t^2} = \frac{1}{\kappa\mu} \nabla^2 \Pi. \qquad 73$$

A solution adapted to the vibrator we have supposed is

$$\Pi = \frac{\Phi}{r} \sin (mr - nt), \qquad 74$$

where r is the distance of the point considered from the origin, and Φ is the maximum moment of the electric doublet.

From this solution the electric and magnetic forces are found by differentiation. In cylindrical coordinates x, ρ, θ the equation becomes

$$\frac{\partial^2 \Pi}{\partial t^2} = \frac{1}{\kappa\mu} \left(\frac{\partial \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Pi}{\partial \rho} \right) \qquad 75$$

since Π is independent of θ. Here $\rho^2 = y^2 + z^2$, and hence if we put now P and R for the axial and radial components of electric force, we must in calculating them from Π use the formulae

$$\left. \begin{aligned} \kappa P &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial P}{\partial \rho} \right), \\ \kappa R &= - \frac{\partial^2 \Pi}{\partial x \partial \rho} \end{aligned} \right\}. \qquad 76$$

We take the meridian plane as plane of x, z, so that the magnetic force H which is at right angles to the meridian plane is identical with β. Thus

$$H = - \frac{\partial^2 \Pi}{\partial t \partial \rho}. \qquad 77$$

The fully worked out results of this solution are very interesting, but, as they do not involve any applications of Bessel functions, we do not consider them in detail. We have referred to them inasmuch as the case of the propagation of waves along a wire, for the solution of which the use of Bessel functions is requisite, may be very instructively compared with this simple case, from which it may be regarded as built up.

In the problem of the wire we have Π at each point of the medium close to the surface of the conductor a simple

harmonic function of the distance of the point from a chosen origin. We shall suppose that the wire is very thin and lies along the axis of x, and is infinitely extended in at least one way so that there is no reflection to be taken into account.

Hence at any point just outside the surface

$$\Pi = A \sin(mx - nt + \epsilon).$$

If we exclude any damping out of the wave or change of form we see that A cannot involve x or t; it is therefore a function of ρ. Thus

$$\Pi = f(\rho) \sin(mx - nt + \epsilon). \quad 78$$

Substitution in the differential equation which holds for the medium gives for f the equation

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} - (m^2 - n^2 \kappa\mu) f = 0. \quad 79$$

Here n^2/m^2 is the square of the velocity of propagation. We shall denote $m^2 - n^2\kappa\mu$ by p^2 and suppose that p^2 is positive, that is that the velocity of propagation is less than that of free propagation in the dielectric. We have therefore instead of 79

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} - p^2 f = 0.$$

This is satisfied by $J_0(ip\rho)$ and by $G_0(ip\rho)$ where $p\rho$ is real. The latter solution only is applicable outside the wire, as f must be zero at infinity. We have therefore in the insulating medium

$$\Pi = 2CG_0(ip\rho) \sin(mx - nt + \epsilon), \quad 80$$

where C and ϵ are constants.

Now by 140, p. 67 above, this solution may be written

$$\Pi = 2C\left\{\int_0^\infty \cos(p\rho \sinh\phi)\, d\phi\right\} \sin(mx - nt + \epsilon).$$

Putting $\rho \sinh\phi = \xi$, we get

$$\Pi = 2C \int_0^\infty \frac{\cos p\xi}{\sqrt{\rho^2 + \xi^2}}\, d\xi \,.\, \sin(mx - nt + \epsilon),$$

or

$$\Pi = C \int_{-\infty}^{+\infty} \frac{\cos p\xi}{\sqrt{\rho^2 + \xi^2}}\, d\xi \,.\, \sin(mx - nt + \epsilon). \quad 81$$

This result may be compared with that obtained above, 74, p. 162, from which it is of course capable of being derived.

When $ip\rho$ is small, we have

$$G_0(ip\rho) = -\log \frac{e^{\gamma} ip\rho}{2},$$

so that neglecting the imaginary part we have

$$G_0(ip\rho) = -\left(\gamma + \log \frac{p\rho}{2}\right).$$

Hence at the surface of the wire

$$\Pi = -2C\left(\gamma + \log \frac{p\rho}{2}\right) \sin(mx - nt + \epsilon). \qquad 82$$

If $p = 0$, that is if the velocity of propagation is that of light, the solution is

$$\Pi = C' \log \rho \sin(mx - nt + \epsilon), \qquad 83$$

as may easily be verified by solving directly for this particular case.

In all cases the wave at any instant in the wire may be divided up into half wave-lengths, such that for each lines of force start out from the wire and return in closed curves which do not intersect, and are symmetrically arranged round the wire. The direction of the force in the curves is reversed for each successive half-wave.

When $p = 0$, the electric force, as may very easily be seen, is normal to the wire, and each curve then consists of a pair of parallel lines, one passing out straight to infinity, the other returning to the wire.

CHAPTER XIV.

DIFFRACTION OF LIGHT.

Case of Symmetry round an Axis.

THE problem here considered is the diffraction produced by a small circular opening in a screen on which falls light propagated in spherical waves from a point source. We take as the axis of symmetry the line drawn from the source to the centre of the opening; and it is required to find the intensity of illumination at any point P of a plane screen parallel to the plane of the opening, and at a fixed distance from the latter.

Let the distance of any point of the edge of the orifice from the source be a, and consider the portion of the wave-front of radius a which fills the orifice. If the angular polar distance of an element of this part of the wave-front be θ, and its longitude be ϕ, the area of the element may be written $a^2 \sin\theta d\theta d\phi$. Putting ξ for the distance of this element from the point, P, of the screen at which the illumination is to be found, regarding the element as a secondary source of light, and using the ordinary fundamental formula, we obtain for the disturbance (displacement or velocity of an ether particle) produced at P by this source the expression

$$\frac{a \sin\theta d\theta d\phi}{\lambda\xi} \sin(m\xi - nt),$$

where $m = 2\pi/\lambda$, $n = 2\pi/T$, λ and T being the length and period of the wave.

Thus, if the angular polar distance of the edge of the orifice be θ_1, the whole disturbance at P is

$$\frac{a}{\lambda}\int_0^{2\pi}\int_0^{\theta_1} \frac{1}{\xi} \sin\theta \sin(m\xi - nt)\, d\theta d\phi.$$

Let ζ be the distance of P from the axis of symmetry, and b the distance of the screen from the nearest point or pole of the spherical wave of radius a, so that the distance of the screen from the element is $a(1-\cos\theta)+b$. Because of the symmetry of the illumination we may suppose without loss of generality that the longitude of the point P is zero. Then the distance ξ from the element to P is given by

$$\xi^2 = \{b + a(1-\cos\theta)\}^2 + (a\sin\theta - \zeta\cos\phi)^2 + \zeta^2\sin^2\phi.$$

Since θ is small this reduces to

$$\xi^2 = b^2 + 4a(a+b)\sin^2\frac{\theta}{2} - 2a\zeta\sin\theta\cos\phi + \zeta^2,$$

or

$$\xi = b + \frac{2a(a+b)}{b}\sin^2\frac{\theta}{2} - \frac{a\zeta}{b}\sin\theta\cos\phi + \frac{\zeta^2}{2b}.$$

If now we write ρ for $a\sin\theta$, or $a\theta$, we have approximately $\sin^2\frac{1}{2}\theta = \rho^2/4a^2$, so that

$$\xi = b + \frac{\zeta^2}{2b} - \frac{\zeta\rho}{b}\cos\phi + \frac{a+b}{2ab}\rho^2.$$

Hence finally if the opening be of so small radius r, and P be so near the axis that we may substitute $1/b$ for the factor $1/\xi$, we obtain for the total disturbance the expression

$$\frac{1}{ab\lambda}\int_0^{2\pi}\int_0^r \sin\left\{m\left(b + \frac{\zeta^2}{2b} - \frac{\zeta\rho}{b}\cos\phi + \frac{a+b}{2ab}\rho^2\right) - nt\right\}\rho d\rho d\phi.$$

Separating now those terms of the argument within the large brackets which do not depend upon ρ from the others, and denoting them by ϖ, so that

$$\varpi = m\left(b + \frac{\zeta^2}{2b}\right) - nt,$$

we may write the expression in the form

$$\frac{1}{ab\lambda}\int_0^{2\pi}\int_0^r \sin(\varpi + \chi)\rho d\rho d\phi,$$

or

$$\frac{1}{ab\lambda}(C\sin\varpi + S\cos\varpi),$$

where

$$C = \int_0^{2\pi} \int_0^r \cos \frac{2\pi}{\lambda} \left(\frac{a+b}{2ab} \rho^2 - \frac{\zeta}{b} \rho \cos \phi \right) \rho d\rho d\phi,$$

$$S = \int_0^{2\pi} \int_0^r \sin \frac{2\pi}{\lambda} \left(\frac{a+b}{2ab} \rho^2 - \frac{\zeta}{b} \rho \cos \phi \right) \rho d\rho d\phi.$$

The intensity of illumination at P is thus proportional to

$$\frac{1}{a^2 b^2 \lambda^2} (C^2 + S^2),$$

and it only remains to calculate the integrals C and S. This can be done by the following process due to Lommel *, depending upon the properties of Bessel Functions.

Changing the order of integration in C we have

$$C = \int_0^r \left\{ \int_0^{2\pi} \cos \frac{2\pi}{\lambda} \left(\frac{a+b}{2ab} \rho^2 - \frac{\zeta}{b} \rho \cos \phi \right) d\phi \right\} \rho d\rho.$$

Now considering the inner integral and writing

$$\frac{2\pi}{\lambda} \frac{a+b}{2ab} \rho^2 = \tfrac{1}{2}\psi, \quad \frac{2\pi}{\lambda} \frac{\zeta}{b} \rho = x,$$

we have

$$\int_0^{2\pi} \cos \frac{2\pi}{\lambda} \left(\frac{a+b}{2ab} \rho^2 - \frac{\zeta}{b} \rho \cos \phi \right) d\phi$$

$$= \int_0^{2\pi} \cos (\tfrac{1}{2}\psi - x \cos \phi) \, d\phi$$

$$= \cos \tfrac{1}{2}\psi \int_0^{2\pi} \cos (x \cos \phi) \, d\phi,$$

since

$$\sin \tfrac{1}{2}\psi \int_0^{2\pi} \sin (x \cos \phi) \, d\phi = 0.$$

But

$$\cos \tfrac{1}{2}\psi \int_0^{2\pi} \cos (x \cos \phi) \, d\phi = 2 \cos \tfrac{1}{2}\psi \int_0^{\pi} \cos (x \cos \phi) \, d\phi$$
$$= 2\pi \cos \tfrac{1}{2}\psi \, . \, J_0(x),$$

by 44, p. 18 above.

Hence

$$C = \frac{b^2 \lambda^2}{2\pi \zeta^2} \int_0^z \cos \tfrac{1}{2}\psi \, . \, x J_0(x) \, dx, \qquad \text{I}$$

where z denotes the value of x when $\rho = r$.

* *Abh. d. k. Bayer. Akad. d. Wissensch.* xv. 1886.

Similarly we can show that

$$S = \frac{b^2\lambda^2}{2\pi\zeta^2}\int_0^z \sin \tfrac{1}{2}\psi \, . \, xJ_0(x)\, dx. \qquad 2$$

These integrals can be expanded in series of Bessel Functions in the following manner. First multiplying 19, p. 13 above, by x^n, and rearranging we obtain

$$x^n J_{n-1}(x) = nx^{n-1}J_n(x) + x^n J_n'(x),$$

and hence by integration

$$\int_0^x x^n J_{n-1}(x)\, dx = x^n J_n(x).$$

Integrating by parts and using this result we get

$$\int \cos \tfrac{1}{2}\psi \, . \, xJ_0(x)\, dx = \cos \tfrac{1}{2}\psi \, . \, xJ_1(x) + \frac{\lambda}{2\pi}\frac{a+b}{ab}\frac{b^2}{\zeta^2}\int \sin \tfrac{1}{2}\psi \, . \, x^2 J_1(x)\, dx.$$

The same process may now be repeated on the integral of the second term on the right and so on. Thus putting $\frac{1}{2}y$ for the value of $\frac{1}{2}\psi$ when $x = z$, and writing

$$\left.\begin{aligned} U_1 &= \frac{y}{z}J_1(z) - \left(\frac{y}{z}\right)^3 J_3(z) + \ldots = \Sigma(-1)^n\left(\frac{y}{z}\right)^{2n+1} J_{2n+1}(z), \\ U_2 &= \left(\frac{y}{z}\right)^2 J_2(z) - \left(\frac{y}{z}\right)^4 J_4(z) + \ldots = \Sigma(-1)^n\left(\frac{y}{z}\right)^{2n+2} J_{2n+2}(z) \end{aligned}\right\}, \qquad 3$$

we obtain finally, putting $4\pi^2\zeta^2 r^2/b^2\lambda^2$ for z^2,

$$C = \pi r^2\left\{\frac{\cos \frac{1}{2}y}{\frac{1}{2}y}U_1 + \frac{\sin \frac{1}{2}y}{\frac{1}{2}y}U_2\right\}, \qquad 4$$

$$S = \pi r^2\left\{\frac{\sin \frac{1}{2}y}{\frac{1}{2}y}U_1 - \frac{\cos \frac{1}{2}y}{\frac{1}{2}y}U_2\right\}. \qquad 5$$

The values of C and S can thus be found by evaluating the series U_1, U_2 for the given value of z. This can be done easily by the numerical tables of Bessel Functions given at the end of this volume.

The series U_1, U_2 proceed by ascending powers of y/z. Series proceeding by ascending powers of z/y can easily be found by a process similar to that used above. We begin by performing the partial integration first upon $\cos \frac{1}{2}\psi \, . \, xdx$, and then continuing

the process, making use of the equation

$$\frac{\partial}{\partial x}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x),$$

which is in fact the relation

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x),$$

stated in 16, p. 13, above.

Thus remembering that

$$\tfrac{1}{2}\psi = \frac{\lambda}{2\pi}\frac{b^2}{\zeta^2}\frac{a+b}{2ab}x^2 = \mu x^2, \text{ say},$$

we have as the first step in the process

$$\begin{aligned}
C &= \frac{b^2\lambda^2}{2\pi\zeta^2}\int_0^z J_0(x)\cos\mu x^2 . x dx \\
&= \frac{b^2\lambda^2}{2\pi\zeta^2}\left\{\frac{1}{2\mu}\sin\mu z^2 . J_0(z) + \frac{1}{2\mu}\int_0^z \frac{1}{x} J_1(x)\sin\mu x^2 . x dx\right\} \\
&= \frac{b^2\lambda^2}{2\pi\zeta^2}\left\{\frac{1}{2\mu}\sin\mu z^2 . J_0(z) - \frac{1}{4\mu^2}\frac{1}{z} J_1(z)\cos\mu z^2\right. \\
&\qquad\qquad \left. + \frac{1}{4\mu^2}\frac{1}{2} - \frac{1}{4\mu^2}\int_0^z \frac{1}{x^2} J_2(x)\cos\mu x^2 . x dx\right\}.
\end{aligned}$$

Proceeding in this way we obtain

$$\begin{aligned}
C &= \pi r^2\left[\frac{\sin\frac{1}{2}y}{\frac{1}{2}y}\left\{J_0(z) - \left(\frac{z}{y}\right)^2 J_2(z) + \dots\right\}\right. \\
&\qquad - \frac{\cos\frac{1}{2}y}{\frac{1}{2}y}\left\{\frac{z}{y}J_1(z) - \left(\frac{z}{y}\right)^3 J_3(z) + \dots\right\} \\
&\qquad \left. + \frac{2}{y}\left\{\frac{z^2}{2y} - \frac{1}{3!}\left(\frac{z^2}{2y}\right)^3 + \dots\right\}\right] \\
&= \pi r^2\left\{\frac{2}{y}\sin\frac{z^2}{2y} + \frac{\sin\frac{1}{2}y}{\frac{1}{2}y}V_0 - \frac{\cos\frac{1}{2}y}{\frac{1}{2}y}V_1\right\}, \qquad 6
\end{aligned}$$

where

$$\left.\begin{aligned}
V_0 &= J_0(z) - \left(\frac{z}{y}\right)^2 J_2(z) + \dots = \Sigma(-1)^n\left(\frac{z}{y}\right)^{2n} J_{2n}(z), \\
V_1 &= \frac{z}{y}J_1(z) - \left(\frac{z}{y}\right)^3 J_3(z) + \dots = \Sigma(-1)^n\left(\frac{z}{y}\right)^{2n+1} J_{2n+1}(z)
\end{aligned}\right\}. \qquad 7$$

Similarly we obtain

$$S = \pi r^2\left\{\frac{2}{y}\cos\frac{z^2}{2y} - \frac{\cos\frac{1}{2}y}{\frac{1}{2}y}V_0 - \frac{\sin\frac{1}{2}y}{\frac{1}{2}y}V_1\right\}. \qquad 8$$

Comparing 4 and 5 with 6 and 8 we get

$$U_1 \cos \tfrac{1}{2}y + U_2 \sin \tfrac{1}{2}y = \sin \frac{z^2}{2y} + V_0 \sin \tfrac{1}{2}y - V_1 \cos \tfrac{1}{2}y,$$

$$U_1 \sin \tfrac{1}{2}y - U_2 \cos \tfrac{1}{2}y = \cos \frac{z^2}{2y} - V_0 \cos \tfrac{1}{2}y - V_1 \sin \tfrac{1}{2}y,$$

which give

$$\left.\begin{aligned} U_1 + V_1 &= \sin \tfrac{1}{2}\left(y + \frac{z^2}{y}\right), \\ -U_2 + V_0 &= \cos \tfrac{1}{2}\left(y + \frac{z^2}{y}\right) \end{aligned}\right\}. \qquad 9$$

Squaring 4 and 5 and 6 and 8 we obtain equivalent expressions for the intensity of illumination at the point P on the screen; thus if $\pi r^2 = 1$

$$\begin{aligned} \frac{1}{a^2b^2\lambda^2}(C^2 + S^2) &= \frac{1}{a^2b^2\lambda^2}\left(\frac{2}{y}\right)^2 (U_1^2 + U_2^2) \\ &= \frac{1}{a^2b^2\lambda^2}\left(\frac{2}{y}\right)^2 \left\{1 + V_0^2 + V_1^2 - 2V_0 \cos \tfrac{1}{2}\left(y + \frac{z^2}{y}\right)\right. \\ &\qquad \left. - 2V_1 \sin \tfrac{1}{2}\left(y + \frac{z^2}{y}\right)\right\}. \end{aligned} \qquad 10$$

The calculation of these U and V functions by means of tables of Bessel Functions will be facilitated by taking advantage of certain properties which they possess. We follow Lommel in the following short discussion of these properties, adopting however a somewhat different analysis.

Consider the more general functions

$$\begin{aligned} U_n &= \left(\frac{y}{z}\right)^n J_n(z) - \left(\frac{y}{z}\right)^{n+2} J_{n+2}(z) + \dots \\ &= \Sigma (-1)^p \left(\frac{y}{z}\right)^{n+2p} J_{n+2p}(z), \end{aligned} \qquad 11$$

$$\begin{aligned} V_n &= \left(\frac{z}{y}\right)^n J_n(z) - \left(\frac{z}{y}\right)^{n+2} J_{n+2}(z) + \dots \\ &= \Sigma (-1)^p \left(\frac{z}{y}\right)^{n+2p} J_{n+2p}(z), \end{aligned} \qquad 12$$

where n may be any positive or negative integer.

First of all it is clear that the series are convergent for all values of y and z. Now if in 75 p. 29 above we put $x = n = 0$, we get

$$1 = J_0^2(z) + 2J_1^2(z) + 2J_2^2(z) + \dots.$$

Hence we see that since $J_0(z) < 1$ each of the other Bessel Functions must be less than $1/\sqrt{2}$. It follows that if $y/z < 1$ the series for U_n is more convergent than the geometric series

$$\Sigma \left(\frac{y}{z}\right)^{n+2p},$$

and if $z/y < 1$, V_n is more convergent than the geometric series

$$\Sigma \left(\frac{z}{y}\right)^{n+2p}.$$

It is therefore more convenient in the former case to use U_n, in the latter to use V_n for purposes of calculation.

If $y = z$

$$U_0 = V_0 = J_0 - J_2 + J_4 - \dots$$
$$U_1 = V_1 = J_1 - J_3 + J_5 - \dots$$

But putting in 42 and 43, p. 18 above, $\phi = 0$, we find

$$\cos z = J_0(z) - 2J_2(z) + 2J_4(z) - \dots$$
$$\sin z = 2J_1(z) - 2J_3(z) + 2J_5(z) - \dots$$

Therefore when $z = y$

$$U_0 = V_0 = \tfrac{1}{2}\{J_0(z) + \cos z\},$$
$$U_1 = V_1 = \tfrac{1}{2}\sin z,$$
$$U_2 = V_2 = \tfrac{1}{2}\{J_0(z) - \cos z\},$$

and generally

$$\left.\begin{aligned} U_{2n} = V_{2n} &= \frac{(-1)^n}{2}\{J_0(z) + \cos z\} - \sum_{p=0}^{p=n-1}(-1)^{n+p} J_{2p}(z), \\ U_{2n+1} = V_{2n+1} &= \frac{(-1)^n}{2}\sin z - \sum_{p=0}^{p=n-1}(-1)^{n+p} J_{2p+1}(z) \end{aligned}\right\}. \qquad 13$$

Returning now to 11 and 12 we easily find

$$\left.\begin{aligned} U_n + U_{n+2} &= \left(\frac{y}{z}\right)^n J_n(z), \\ V_n + V_{n+2} &= \left(\frac{z}{y}\right)^n J_n(z) \end{aligned}\right\}, \qquad 14$$

and therefore

$$z^{2n}(U_n + U_{n+2}) = y^{2n}(V_n + V_{n+2}). \qquad 15$$

Also since $J_{-n}(z) = (-1)^n J_n(z)$, we find, putting $-n$ for n in the second and first of 14 successively, and also $z = y$

$$U_n + U_{n+2} = (-1)^n (V_{-n} + V_{-n+2}),$$
$$V_n + V_{n+2} = (-1)^n (U_{-n} + U_{-n+2}).$$

Differentiating 11, we find

$$\frac{\partial U_n}{\partial z} = -\frac{n}{z}\left(\frac{y}{z}\right)^n J_n(z) + \frac{n+2}{z}\left(\frac{y}{z}\right)^{n+2} J_{n+2}(z) - \dots$$
$$+ \left(\frac{y}{z}\right)^n J_n'(z) - \left(\frac{y}{z}\right)^{n+2} J_{n+2}'(z) + \dots.$$

Using in the second line of this result the relation [16, p. 13 above]

$$J_n'(z) = \frac{n}{z} J_n(z) - J_{n+1}(z)$$

we get

$$\frac{\partial U_n}{\partial z} = -\left(\frac{y}{z}\right)^n J_{n+1}(z) + \left(\frac{y}{z}\right)^{n+2} J_{n+3}(z) - \dots$$
$$= -\frac{z}{y} U_{n+1}. \qquad 16$$

This gives by successive differentiation the equation

$$\frac{\partial^m U_n}{\partial z^m} = -\frac{m-1}{y}\frac{\partial^{m-2}}{\partial z^{m-2}} U_{n+1} - \frac{z}{y}\frac{\partial^{m-1}}{\partial z^{m-1}} U_{n+1}. \qquad 17$$

Similarly we obtain by differentiating 12 and using the relation [19, p. 13 above]

$$J_n'(z) = J_{n-1}(z) - \frac{n}{z} J_n(z),$$

$$\frac{\partial V_n}{\partial z} = \frac{z}{y} V_{n-1}, \qquad 18$$

and therefore

$$\frac{\partial^m V_n}{\partial z^m} = \frac{m-1}{y}\frac{\partial^{m-2}}{\partial z^{m-2}} V_{n-1} + \frac{z}{y}\frac{\partial^{m-1}}{\partial z^{m-1}} V_{n-1}. \qquad 19$$

Again differentiating the first of 9, we get

$$\frac{\partial U_1}{\partial z} + \frac{\partial V_1}{\partial z} = \frac{z}{y}\cos\tfrac{1}{2}\left(y + \frac{z^2}{y}\right).$$

But by 16 and 18 this becomes

$$-U_2 + V_0 = \cos\tfrac{1}{2}\left(y + \frac{z^2}{y}\right).$$

Differentiating again we obtain

$$-\frac{\partial U_2}{\partial z} + \frac{\partial V_0}{\partial z} = -\frac{z}{y}\sin\tfrac{1}{2}\left(y + \frac{z^2}{y}\right),$$

or by 16 and 18

$$U_3 + V_{-1} = -\sin\tfrac{1}{2}\left(y + \frac{z^2}{y}\right).$$

By repeating this process it is clear that we shall obtain

$$\left.\begin{aligned} U_{2n+1} + V_{-2n+1} &= (-1)^n \sin \tfrac{1}{2}\left(y + \frac{z^2}{y}\right), \\ -U_{2n+2} + V_{-2n} &= (-1)^n \cos \tfrac{1}{2}\left(y + \frac{z^2}{y}\right) \end{aligned}\right\}. \qquad 20$$

If in these equations we put $n = 0$, we fall back upon 9. Putting in 9 the values of the functions as given in the defining equations 11 and 12 we obtain the theorems

$$\left.\begin{aligned} &\Sigma(-1)^p \left\{\left(\frac{y}{z}\right)^{2p+1} + \left(\frac{z}{y}\right)^{2p+1}\right\} J_{2p+1}(z) = \sin \tfrac{1}{2}\left(y + \frac{z^2}{y}\right), \\ &\Sigma(-1)^p \left\{\left(\frac{y}{z}\right)^{2p+2} + \left(\frac{z}{y}\right)^{2p+2}\right\} J_{2p+2}(z) = J_0(z) - \cos \tfrac{1}{2}\left(y + \frac{z^2}{y}\right) \end{aligned}\right\}, \qquad 21$$

which include the equations

$$\sin z = 2J_1(z) - 2J_3(z) + 2J_5(z) - \dots$$
$$\cos z = J_0(z) - 2J_2(z) + 2J_4(z) - \dots$$

as particular cases, those namely, for which $y = z$.

By Taylor's theorem we have

$$U_n(y, z + h) = U_n + h\frac{\partial U_n}{\partial z} + \frac{h^2}{2!}\frac{\partial^2 U_n}{\partial z^2} + \dots .$$

Calculating the successive differential coefficients by means of 16, and rearranging the terms we obtain

$$\begin{aligned} U_n(y, z + h) &= U_n - \frac{h(2z + h)}{2y} U_{n+1} + \frac{1}{2!}\frac{h^2(2z + h)^2}{(2y)^2} U_{n+2} - \dots \\ &= \Sigma(-1)^p \frac{h^p(2z + h)^p}{p!\,(2y)^p} U_{n+p}. \end{aligned} \qquad 22$$

Similarly we can prove that

$$V_n(y, z + h) = \Sigma \frac{h^p(2z + h)^p}{p!\,(2y)^p} V_{n-p}. \qquad 23$$

These expansions are highly convergent and permit of easy calculation of $U_n(y, z + h)$, $V_n(y, z + h)$. The functions U_{n+1}, U_{n+2}, $U_{n+3}, \dots$, V_{n-1}, $V_{n-2}, \dots$, can be found from U_n, V_n, by using 16 and 18 to calculate U_{n+1}, V_{n-1}, and then deducing the others by successive applications of 14.

Differentiating 11 and 12 with respect to y, and using in the resulting expressions the relation 20, p. 13 above, namely,

$$nJ_n(z) = \tfrac{1}{2}z\, J_{n-1}(z) + \tfrac{1}{2}z\, J_{n+1}(z),$$

we find

$$\left.\begin{aligned}\tfrac{1}{2}z^2 U_{n+1} &= y^2 \frac{\partial U_n}{\partial y} - \tfrac{1}{2}y^2 U_{n-1},\\ \tfrac{1}{2}z^2 V_{n-1} &= -\, y^2 \frac{\partial V_n}{\partial y} - \tfrac{1}{2}y^2 V_{n+1}\end{aligned}\right\}. \qquad 24$$

Now if u be a function of y we have

$$\frac{\partial^m (y^2 u)}{\partial y^m} = (m-1)\, m \frac{\partial^{m-2} u}{\partial y^{m-2}} + 2my \frac{\partial^{m-1} u}{\partial y^{m-1}} + y^2 \frac{\partial^m u}{\partial y^m}.$$

Using this theorem we find by successive differentiation of 24

$$\begin{aligned}\tfrac{1}{2}z^2 \frac{\partial^m U_{n+1}}{\partial y^m} &= y^2 \left(\frac{\partial^{m+1} U_n}{\partial y^{m+1}} - \tfrac{1}{2}\frac{\partial^m U_{n-1}}{\partial y^m}\right)\\ &\quad + 2my \left(\frac{\partial^m U_n}{\partial y^m} - \tfrac{1}{2}\frac{\partial^{m-1} U_{n-1}}{\partial y^{m-1}}\right)\\ &\quad + (m-1)\, m \left(\frac{\partial^{m-1} U_n}{\partial y^{m-1}} - \tfrac{1}{2}\frac{\partial^{m-2} U_{n-1}}{\partial y^{m-2}}\right), \qquad 25\end{aligned}$$

$$\begin{aligned}-\tfrac{1}{2}z^2 \frac{\partial^m V_{n-1}}{\partial y^m} &= y^2 \left(\frac{\partial^{m+1} V_n}{\partial y^{m+1}} + \tfrac{1}{2}\frac{\partial^m V_{n+1}}{\partial y^m}\right)\\ &\quad + 2my \left(\frac{\partial^m V_n}{\partial y^m} + \tfrac{1}{2}\frac{\partial^{m-1} V_{n+1}}{\partial y^{m-1}}\right)\\ &\quad + (m-1)\, m \left(\frac{\partial^{m-1} V_n}{\partial y^{m-1}} + \tfrac{1}{2}\frac{\partial^{m-2} V_{n+1}}{\partial y^{m-2}}\right). \qquad 26\end{aligned}$$

If we consider y as a function of z then

$$\frac{dU_n}{dz} = \frac{\partial U_n}{\partial z} + \frac{\partial U_n}{\partial y}\frac{dy}{dz}.$$

If $y = cz$

$$\frac{dU_n}{dz} = \tfrac{1}{2}\left(cU_{n-1} - \frac{1}{c}\, U_{n+1}\right),$$

by 16 and 24 above. By successive differentiation, and application of this result we obtain

$$\frac{d^2 U_n}{dz^2} = \tfrac{1}{4}\left(c^2 U_{n-2} - 2U_n + \frac{1}{c^2}\, U_{n+2}\right),$$

and generally

$$\frac{d^m U_n}{dz^m} = \frac{1}{2^m}\Sigma(-1)^p \frac{m(m-1)\ldots(m-p+1)}{p!}\, c^{m-2p}\, U_{n-m+2p}. \qquad 27$$

Similarly it can be shown that

$$\frac{d^m V_n}{dz^m} = \frac{1}{2^m} \Sigma (-1)^p \frac{m(m-1)\dots(m-p+1)}{p!} c^{-m+2p} V_{n-m+2p}. \qquad 28$$

The calculation of the differential coefficients can be carried out by these formulae with the assistance of 14 which now become

$$\left.\begin{aligned} U_n + U_{n+2} &= c^n J_n(z), \\ V_n + V_{n+2} &= \frac{1}{c^n} J_n(z) \end{aligned}\right\}. \qquad 29$$

We conclude this analytical discussion with some theorems in which definite integrals involving Bessel Functions are expressed in terms of the U and V functions.

By 1 above we have

$$C = \frac{b^2\lambda^2}{2\pi\zeta^2} \int_0^z \cos \tfrac{1}{2}\psi \,.\, x J_0(x)\, dx.$$

Now let $x = zu$, then

$$\tfrac{1}{2}\psi = \tfrac{1}{2} y \frac{x^2}{z^2} = \tfrac{1}{2} y u^2,$$

therefore since $z^2 = 4\pi^2\zeta^2 r^2/\lambda^2 b^2$

$$C = 2\pi r^2 \int_0^1 \cos(\tfrac{1}{2} y u^2) \,.\, u J_0(zu)\, du. \qquad 30$$

Similarly we obtain

$$S = 2\pi r^2 \int_0^1 \sin(\tfrac{1}{2} y u^2) \,.\, u J_0(zu)\, du. \qquad 31$$

But equations 4 and 5 give

$$C \cos \tfrac{1}{2} y + S \sin \tfrac{1}{2} y = \frac{\pi r^2}{\tfrac{1}{2} y} U_1,$$

$$C \sin \tfrac{1}{2} y - S \cos \tfrac{1}{2} y = \frac{\pi r^2}{\tfrac{1}{2} y} U_2,$$

and these by 30 and 31 give the equations

$$\left.\begin{aligned} \int_0^1 J_0(zu) \,.\, \cos \tfrac{1}{2} y (1-u^2) \,.\, u du &= \frac{1}{y} U_1 \\ \int_0^1 J_0(zu) \,.\, \sin \tfrac{1}{2} y (1-u^2) \,.\, u du &= \frac{1}{y} U_2 \end{aligned}\right\}. \qquad 32$$

Differentiating with respect to z we get, since

$$J_0'(z) = -J_1(z),$$

$$\int_0^1 J_1(zu)\cos\tfrac{1}{2}y(1-u^2)\,.\,u^2du = -\frac{1}{y}\frac{\partial U_1}{\partial z}$$

$$= \frac{z}{y^2}U_2;$$

and similarly

$$\int_0^1 J_1(zu)\sin\tfrac{1}{2}y(1-u^2)\,.\,u^2du = \frac{z}{y^2}U_3.$$

Now if we assume

$$\int_0^1 J_{n-1}(zu)\,.\cos\tfrac{1}{2}y(1-u^2)\,.\,u^n du = \frac{1}{y}\left(\frac{z}{y}\right)^{n-1} U_n \qquad 33$$

and differentiate, making use of the relation

$$J_{n-1}'(zu) = \frac{n-1}{zu}J_{n-1}(zu) - J_n(zu),$$

we easily obtain

$$\int_0^1 J_n(zu)\cos\tfrac{1}{2}y(1-u^2)\,.\,u^{n+1}du = \frac{1}{y}\left(\frac{z}{y}\right)^n U_{n+1}.$$

Thus if the theorem 33 hold for any integral value of n it holds for $n+1$. But as we have seen above it holds for $n=1$, it therefore holds for all integral values of n.

Similarly we obtain

$$\int_0^1 J_{n-2}(zu)\sin\tfrac{1}{2}y(1-u^2)\,u^{n-1}du = \frac{1}{y}\left(\frac{z}{y}\right)^{n-2} U_n. \qquad 34$$

The values of C in 6 and 30 give

$$\int_0^1 \cos\tfrac{1}{2}yu^2\,.\,uJ_0(zu)\,du = \frac{1}{y}\sin\frac{z^2}{2y} + \frac{\sin\frac{1}{2}y}{y}V_0 - \frac{\cos\frac{1}{2}y}{y}V_1. \qquad 35$$

Similarly those of S in 8 and 31 give

$$\int_0^1 \sin\tfrac{1}{2}yu^2\,.\,uJ_0(zu)\,du = \frac{1}{y}\cos\frac{z^2}{2y} - \frac{\cos\frac{1}{2}y}{y}V_0 - \frac{\sin\frac{1}{2}y}{y}V_1. \qquad 36$$

If instead of u we use the variable $\rho\,(=ur)$ where r is the radius of the orifice, and write $y=kr^2$, $z=lr$, we have instead of 35, 36

$$\int_0^r J_0(l\rho)\cos(\tfrac{1}{2}k\rho^2)\,.\,\rho d\rho = \frac{1}{k}\sin\frac{l^2}{2k} + \frac{1}{k}\sin(\tfrac{1}{2}kr^2)\,V_0 - \frac{1}{k}\cos(\tfrac{1}{2}kr^2)\,V_1,$$

$$\int_0^r J_0(l\rho)\sin(\tfrac{1}{2}k\rho^2)\,.\,\rho d\rho = \frac{1}{k}\cos\frac{l^2}{2k} - \frac{1}{k}\cos(\tfrac{1}{2}kr^2)\,V_0 - \frac{1}{k}\sin(\tfrac{1}{2}kr^2)\,V_1.$$

If now r be made infinite while l and k do not vanish, V_0 and V_1 vanish, and we have

$$\left.\begin{aligned}\int_0^\infty J_0(l\rho)\cos(\tfrac{1}{2}k\rho^2)\,.\,\rho d\rho &= \frac{1}{k}\sin\frac{l^2}{2k}\\ \int_0^\infty J_0(l\rho)\sin(\tfrac{1}{2}k\rho^2)\,.\,\rho d\rho &= \frac{1}{k}\cos\frac{l^2}{2k}\end{aligned}\right\}, \qquad 37$$

formulæ which will be found useful in what follows. They are special cases of more general theorems which can easily be obtained by successive differentiation.

We come now to the application of these results to the problem stated above. Of this problem there are two cases which may be distinguished, (1) that in which $y = 0$, (2) that in which y does not vanish. The first case is that of Fraunhofer's diffraction phenomena, and has received much attention. We shall consider it specially here, and afterwards pass on to the more general case (2).

When $y = 0$, either $a = \infty$ and $b = \infty$, or $a = -b$. In the former case the wave incident on the orifice is plane, and the parallel screen on which the light from the orifice falls is at a very great distance from the orifice. This arrangement, as Lommel points out, is realised when the interference phenomena are observed with a spectrometer, the telescope and collimator of which are adjusted for parallel rays. The orifice is placed between the collimator and the telescope at right angles to the parallel beam produced by the former.

When $a = -b$, a may be either positive or negative. When a is negative the orifice is to be supposed illuminated by light converging *to* the point-source, and the screen is there situated with its plane at right angles to the axis of symmetry. This can be realised at once by producing a converging beam of light by means of a convex lens, and then introducing the orifice between the lens and the screen, which now coincides with the focal plane of the lens.

When a is positive, and therefore b negative, the light-wave falls on the orifice, with its front convex towards the direction of propagation. The interference is then to be considered as produced on a screen passing through the source, and at right angles to the line joining the source with the centre of the orifice.

This case can be virtually realised by receiving the light from the opening by an eye focused on the source. The diffraction pattern is then produced on the retina. Or, a convex lens may be placed at a greater distance from the source than the principal focal distance of the lens, so as to receive the light after having passed the orifice, and the screen in the focal plane of the lens.

The screen may be examined by the naked eye or through a magnifying lens. If a magnifying lens is used the arrangement is equivalent to a telescope focused upon the point-source, with the opening in front of the object-glass. This is Fraunhofer's arrangement; and we shall obtain the theory of the phenomena observed by him if we put $y = 0$ in the above theoretical investigation.

Putting $y = 0$ in 3, we have

$$\frac{2}{y} U_1 = \frac{2}{z} J_1(z), \quad \frac{2}{y} U_2 = 0,$$

so that, writing M^2 for $C^2 + S^2$, we obtain

$$M^2 = \left\{\frac{2}{z} J_1(z)\right\}^2. \tag{38}$$

Airy gave* for the same quantity the expression, in the present notation,

$$\left\{1 - \frac{z^2}{2.4} + \frac{z^4}{2.4.4.6} - \frac{z^6}{2.4.4.6.6.8} + \dots\right\}^2,$$

which is simply the quantity on the right of 38.

By means of Tables of Bessel Functions the value of M can be found with the greatest ease, by simply doubling the value of J_1 for any given argument, and dividing the result by the argument. The result is shown graphically in the adjoining diagram.

The maxima of light intensity are at those points for which $J_1(z)/z$ is a maximum or a minimum. The minima are those points for which $J_1(z) = 0$. Now when $J_1(z)/z$ is a maximum or a minimum

$$\frac{\partial}{\partial z}\left\{\frac{1}{z} J_1(z)\right\} = 0.$$

* *Camb. Phil. Trans.* p. 283, 1834.

But

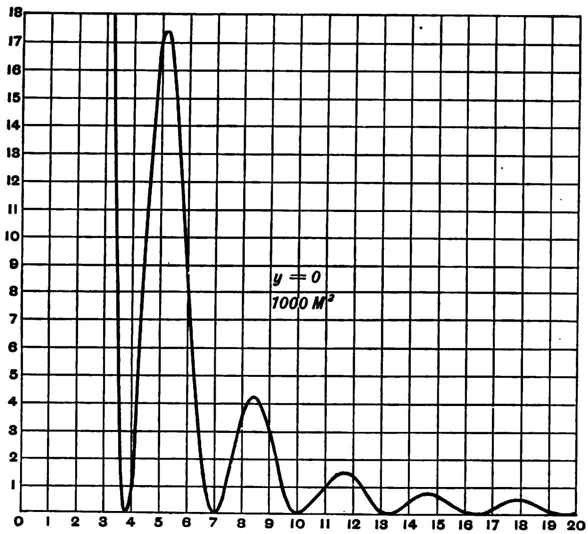
$$\frac{1}{z}J_1'(z) - \frac{J_1(z)}{z^2} = -\frac{1}{z}J_2(z),$$

so that the condition becomes

$$J_2(z) = 0,$$

which (20, p. 13 above) is equivalent to

$$\frac{2}{z}J_1(z) - J_0(z) = 0.$$



Thus when the maxima and minima values of $2J_1(z)/z$ have been calculated, their accuracy may be checked by observing whether they are also the values of $J_0(z)$ given in the Table for the same arguments. Or the arguments, which are the roots of $J_2(z) = 0$, having been obtained, the corresponding values of $2J_1(z)/z$ are given by the Table of $J_0(z)$ either directly or by interpolation.

Places of maximum intensity alternate with places at which the intensity is zero, the light being supposed of definite wave-length, and therefore monochromatic. The roots of $J_1(z) = 0$, which are the values of z for which the intensity is zero, can be calculated by the formula given at p. 49 above (Chap. V.). It is there shown that the approximate value of the large roots of $J_1(z) = 0$,

is $(m+\frac{1}{4})\pi$, and of $J_2(z)=0$, is $(m+\frac{3}{4})\pi$, where m is the number of the root. Hence for great values of z the difference between the values of z for successive maxima or minima is approximately π, and the difference for a zero and the next following maximum is $\frac{1}{2}\pi$. The rings are thus ultimately equidistant.

The difference of path of the rays from opposite extremities of a diameter of the orifice to the point P is $2r\tan^{-1}\zeta/b$, that is $2r\zeta/b$ or $\lambda z/\pi$. The distance in wave-lengths is therefore z/π.

The following Table gives the values of z corresponding to maximum and zero values of $2z^{-1}J_1(z)$, which are contained in col. 2. Col. 3 contains the corresponding values of M^2.

z (roots of $J_2(z)=0$)	$2z^{-1}J_1(z)$	M^2
0	+1	1
3·831706	0	0
5·135630	−0·132279	0·017498
7·015587	0	0
8·417236	+0·064482	0·004158
10·173467	0	0
11·619857	−0·040008	0·001601
13·323690	0	0
14·795938	+0·027919	0·000779
16·470631	0	0
17·959820	−0·020905	0·000437
19·615861	0	0

For large values of z the semi-convergent expansion of $J_1(z)$ (p. 40 above) is available. As z increases this expansion gives more and more approximately

$$\frac{2}{z}J_1(z)=\frac{2}{z}\sqrt{\frac{2}{\pi z}}\sin(z-\tfrac{1}{4}\pi),$$

and therefore

$$M^2=\frac{8}{\pi z^3}\sin^2(z-\tfrac{1}{4}\pi).$$

As the value of z approaches $(m+\frac{3}{4})\pi$ that of $\sin^2(z-\frac{1}{4}\pi)$ approaches 1, and so the ultimate value of M^2z^3 is $8/\pi$.

The whole light received within a circle of radius z is proportional to

$$\int_0^z M^2 z dz = 4\int_0^z z^{-1} J_1^2(z)\, dz.$$

But

$$\frac{J_1^2(z)}{z} = \{J_0(z) - J_1'(z)\} J_1(z)$$

$$= - J_0(z) J_0'(z) - J_1(z) J_1'(z).$$

Hence

$$\int_0^z M^2 z dz = 2\{1 - J_0^2(z) - J_1^2(z)\}. \qquad 39$$

If z is made infinite the expression in the brackets becomes 1. Hence, as has been pointed out by Lord Rayleigh*, the fraction of the total illumination outside any value of z is $J_0^2(z) + J_1^2(z)$. But at a dark ring $J_1(z) = 0$, so that the fraction of the whole light outside any dark ring is $J_0^2(z)$. The values of this fraction for the successive roots of $J_1(z) = 0$, are approximately ·161, ·090, ·062, ·047, ..., so that more than $\frac{9}{10}$ of the whole light is received within the second dark ring.

In the more general case of diffraction, contemplated by Fresnel, y is not zero, and we have

$$M^2 = \left(\frac{2}{y}\right)^2 (U_1^2 + U_2^2), \qquad 40$$

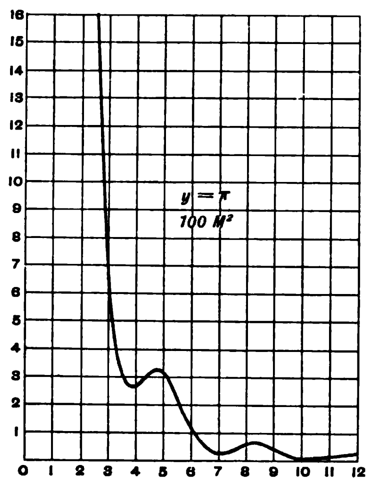
and U_1, U_2 can be calculated by the formulæ given above from the Tables of Bessel Functions at the end of the present volume, equation 3 being used if $z > y$, and 9, with the expansions of V_0 and V_1 if $z < y$.

The maximum and minimum values of M^2 are given in the Table below for the values of y stated. We also give here some diagrams showing the forms of the intensity curve for the same values of y. The curves are drawn with values of z as abscissæ, and of M^2 as ordinates.

* *Phil. Mag.*, March, 1881; or 'Wave Theory of Light,' *Encyc. Brit.*, 9th Edition, p. 433.

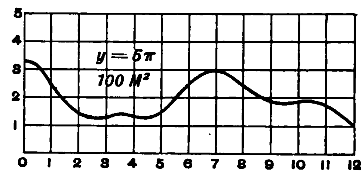
$y=\pi.$

z	$\frac{2}{y}U_1$	$\frac{2}{y}U_2$	M^2	
3·831706	− 0·122609	+ 0·106159	0·026305	Min.
4·715350	− 0·178789	0	0·031966	Max.
7·015587	+ 0·013239	− 0·040631	0·001826	Min.
8·306007	+ 0·074093	0	0·005490	Max.
10·173467	− 0·002313	+ 0·016225	0·000269	Min.
11·578479	− 0·043104	0	0·001858	Max.



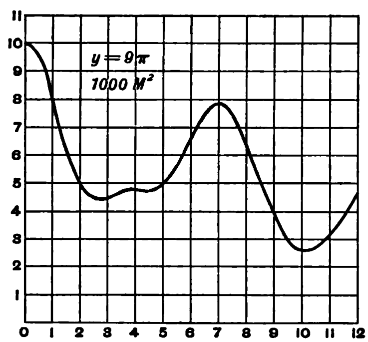
$y=5\pi.$

z	$\frac{2}{y}U_1$	$\frac{2}{y}U_2$	M^2	
3·030827	+ 0·114161	0	0·013033	Min.
3·625773	+ 0·114593	0	0·013132	Max.
3·831706	+ 0·114492	+ 0·004496	0·013128	Min.
7·015587	− 0·002099	+ 0·173617	0·030147	Max.
9·440724	− 0·134688	0	0·018141	Min.
10·173467	− 0·118330	− 0·067421	0·018548	Max.



$y = 9\pi.$

z	$\frac{2}{y} U_1$	$\frac{2}{y} U_2$	M^2	
2·649454	+ 0·067178	0	0·004513	Min.
3·831706	+ 0·068485	− 0·010782	0·004806	Max.
4·431978	+ 0·068964	0	0·004756	Min.
7·015587	+ 0·045384	+ 0·076624	0·007931	Max.
10·173467	− 0·017711	+ 0·048204	0·002637	Min.



The maximum and minimum values of M^2 are those for which

$$\frac{\partial M^2}{\partial z} = 0.$$

But

$$\frac{\partial M^2}{\partial z} = 2\left(\frac{2}{y}\right)^2 \left(U_1 \frac{\partial U_1}{\partial z} + U_2 \frac{\partial U_2}{\partial z}\right)$$

$$= -\left(\frac{2}{y}\right)^2 \frac{2z}{y}\, U_2 (U_1 + U_3)$$

$$= -2\left(\frac{2}{y}\right)^2 J_1(z)\, U_2. \qquad 41$$

Hence a maximum or a minimum is obtained when either

$$\left.\begin{aligned} J_1(z) &= -J_0'(z) = 0 \\ \text{or} \quad U_2 &= -\frac{y}{z}\frac{\partial U_1}{\partial z} = 0 \end{aligned}\right\}. \qquad 42$$

Thus a value of z which gives a maximum or minimum of $J_0(z)$ or of $U_1(z)$ gives either a maximum or minimum of M^2. The roots of $J_1(z) = 0$ are given at the end of this treatise, and

are values of z which give a maximum or minimum of illumination. The values of $2U_1/y$, $2U_2/y$ which correspond to these values of z, are obtained by interpolation from those of U_1, U_2 or V_0, V_1 for the values of z for which Tables of Bessel Functions are available. The formulæ of interpolation are 22, 23 above.

The maxima and minima which arise through the vanishing of U_2 are found in a similar manner. Supposing it is required to find the roots of U_2, the tabular value of $U_2(z)$ nearest to a zero value is taken, and the value of $z+h$ which causes U_2 to vanish is found by means of the expression on the right of 22 equated to zero, with 2 put for n, that is from the equation

$$U_2(z) - \frac{h(2z+h)}{2y} U_3(z) + \frac{1}{2!}\frac{h^2(2z+h)^2}{(2y)^2} U_4(z) - \ldots = 0. \quad 43$$

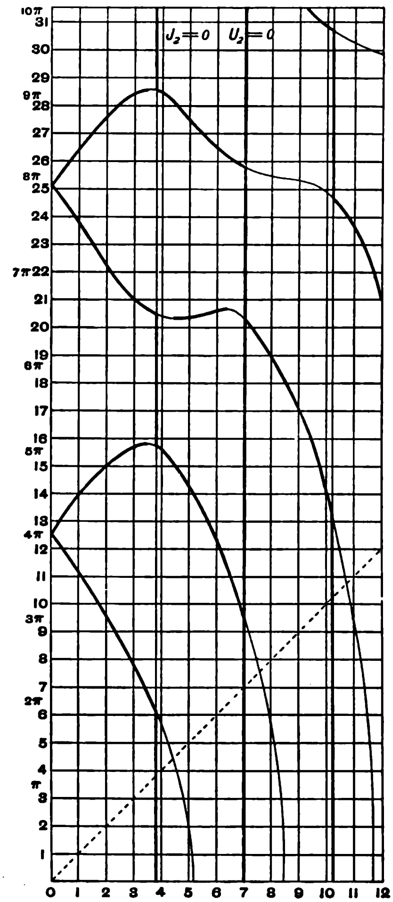
Since the series is very convergent only a few terms need be retained; and the value of $h(2z+h)/2y$ found, and therefore that of h.

Values of z which render $U_2 = 0$, being thus found, those of $2U_1/y$ for the same arguments are calculated. The squares of these are the values of M^2 which correspond to the roots of $U_2 = 0$. Elaborate Tables, each accompanied by a graphical representation of the results, are given by Lommel in his memoir. The short Tables with the illustrative diagrams given above will serve as a specimen.

We conclude the discussion of this case of diffraction with an account of an interesting graphical method of finding, for different values of y, the values of z which give maxima or minima. This is shown in the next diagram. The axis of ordinates is that of y, and the axis of abscissæ that of z. Lines parallel to the axis of y are drawn for the values of z which satisfy $J_1(z) = 0$. These are called the lines $J_1(z) = 0$. On the same diagram are drawn the curves $U_2/y^2 = 0$. These are transcendental curves having double points on the axis $z = 0$, as will be seen from the short discussion below.

Let now the edge of a sheet of paper be kept parallel to the axis of z and be moved along the diagram from bottom to top. It will intersect all the curves. The distances from the axis of y along the edge of the paper in any of its positions to the points of intersection are values of z, for the value of y for that position,

which satisfy 41; and are therefore values of z which with that value of y give maximum or minimum values of M^2.



The equation of the curve $U_2 = 0$ differentiated gives

$$\frac{\partial U_2}{\partial z} + \frac{\partial U_2}{\partial y}\frac{dy}{dz} = 0,$$

that is, since by 16 and 24 above,

$$\partial U_2/\partial z = -U_3 \,.\, z/y,$$

$$\partial U_2/\partial y = \tfrac{1}{2}U_1 + \tfrac{1}{2}U_3 \,.\, (z/y)^2;$$

$$\frac{dy}{dz} = 2\frac{z}{y}\frac{U_3}{U_1 + \left(\frac{z}{y}\right)^2 U_3}. \qquad 44$$

When $z=0$, that is where the curve meets the axis of y, $V_0=1$, $V_1=0$, so that by 9,

$$U_1=\sin\tfrac{1}{2}y,$$
$$U_2=1-\cos\tfrac{1}{2}y.$$

Thus $U_2=0$, whenever $\cos\frac{1}{2}y=1$, that is when $\frac{1}{2}y=2m\pi$, or $y=4m\pi$. The curve $U_2=0$ therefore meets the axis of y at every multiple of 4π.

But this value of y makes U_1 and likewise

$$U_2\,.\,z/y\,\{=J_1(z)-U_1\,.\,z/y\}$$

zero, so that

$$\frac{\partial U_2}{\partial z}=0,\ \frac{\partial U_2}{\partial y}=0.$$

The value of dy/dz is therefore indeterminate at the points on the axis of y, that is each point in which the curve meets the axis of y is a double point.

If y', z' be current coordinates the equation of the pair of tangents at a double point is

$$z'^2\frac{\partial^2 U_2}{\partial z^2}+2z'(y'-y)\frac{\partial^2 U_2}{\partial y\partial z}+(y'-y)^2\frac{\partial^2 U_2}{\partial y^2}=0. \qquad 45$$

It is very easy to verify by differentiation and use of the properties of the functions that when $z=0$ and $y=4m\pi$,

$$\frac{\partial^2 U_2}{\partial z^2}=-\tfrac{1}{2},\ \frac{\partial^2 U_2}{\partial y\partial z}=0,\ \frac{\partial^2 U_2}{\partial y^2}=\tfrac{1}{4},$$

so that the equation of the tangents reduces to

$$(y'-4m\pi)^2=2z'^2.$$

Thus the equations of the tangents are

$$y'-4m\pi=\sqrt{2}z',\ y'-4m\pi=-\sqrt{2}z',$$

and their inclinations to the axis of z are given by

$$\tan\phi=\pm\sqrt{2},$$

that is $$\phi=\pm 54°\,44'\,8''\cdot 2.$$

Where the curve meets the axis of z we may regard $U_2=0$ as equivalent to the two equations $y^2=0$, $U_2/y^2=0$, so that the curve $U_2=0$ splits into two straight lines coincident with the axis of z, and the curve represented by

$$\frac{1}{y^2}U_2=0.$$

We have by 3,

$$\frac{\partial}{\partial y}\left(\frac{1}{y^2}U_2\right) = -2\frac{y}{z^4}J_4(z) + 4\frac{y^3}{z^6}J_6(z) - \ldots.$$

Hence for the last curve

$$\frac{dy}{dz} = -\frac{\dfrac{\partial}{\partial z}\left(\dfrac{1}{y^2}U_2\right)}{\dfrac{\partial}{\partial y}\left(\dfrac{1}{y^2}U_2\right)} = \frac{\dfrac{z}{y^3}U_3}{0} = \infty,$$

when $y = 0$. Thus the branches of the curve $U_2/y^2 = 0$ cut the axis of abscissæ at right angles, as shown in the diagram on p. 185 above. We shall see below that this intersection takes place at points satisfying the equation $J_2(z) = 0$. Thus the curves U_2/y^2, when $y = 0$, touch the curves $J_2(z) = 0$.

It will be seen from the curves in the diagram that the value of dy/dz is negative so long at least as $y < z$, that is, as we shall see, within the region of the curve corresponding to the geometrical shadow. But at points along a line $J_1(z) = 0$,

$$\frac{dy}{dz} = -\frac{2\dfrac{z}{y}}{1-\left(\dfrac{z}{y}\right)^2},$$

and is positive so long as $z > y$, that is also within the region of the geometrical shadow. Hence no intersection of a line $J_1(z) = 0$ with the other curves can exist in the region of the diagram corresponding to the geometrical shadow.

To settle where the maxima and minima are we have to calculate $\partial^2 M^2/\partial z^2$. Now

$$\begin{aligned}\frac{\partial^2 M^2}{\partial z^2} &= -2\left(\frac{2}{y}\right)^2 \frac{\partial}{\partial z}(J_1 U_2)\\ &= 2\left(\frac{2}{y}\right)^2 \left\{\frac{1}{z}J_1 U_2 - J_0 U_2 + J_1\left(J_1 - \frac{z}{y}U_1\right)\right\}. \qquad 46\end{aligned}$$

Thus considering first points upon the lines $J_1 = 0$, we have a maximum or a minimum according as $J_0 U_2$ is positive or negative.

On the other hand, when $U_2 = 0$ the points on the curves $U_2 = 0$ are maxima or minima according as $J_1 U_3 z/y$ $\{= J_1(J_1 - U_1 z/y)\}$ is negative or positive, or as

$$J_1^2 < \text{ or } > \frac{z}{y}U_1 J_1.$$

Calculating $\partial^3 M^2/\partial z^3$ we see that this does not vanish for points satisfying the equations $J_1(z)=0$, $U_2=0$, that is wherever a line $J_1(z)=0$ and a curve $U_2=0$, intersect there is a point of inflexion of the curve of intensity, drawn with M^2 as ordinates and values of z as abscissæ.

It follows by the statement above as to the inclination of the curve within the region corresponding to the geometrical shadow, that within that region there can be no point of inflexion on the intensity-curve.

Also, as can easily be verified, there are points of inflexion of the intensity-curve, wherever the curve $U_2/y^2=0$ has a maximum or minimum ordinate.

Referring now to the diagram, p. 185, we can see how to indicate the points where there are maxima and minima. For pass along a line $J_1=0$ until a branch of the curve $U_2/y^2=0$ is crossed. Here clearly U_2 changes sign, while $J_0(z)$ does not. Thus $J_0(z)\,U_2$ changes sign, and so all points of a portion of a curve $J_1(z)=0$, intercepted between branches of the other curve, give maxima, or give minima, according to the number of branches of the latter which have been crossed to reach that portion by proceeding along $J_1(z)=0$ from the axis of z. $J_0(z)\,U_2$ is negative for the first portion, positive for the second, and so on, the number of crossings being 0, 1, 2, ...

If we pass along a curve $U_2/y^2=0$ and cross $J_1(z)=0$, then $J_1(z)$ changes sign, but not so U_2; for by 14 when $J_1(z)=0$, $U_3=-U_1$, and U_1 is a maximum or a minimum, since $U_2=0$. But it must be further noticed that when for a branch of the curve $U_2/y^2=0$ the value of dy/dz is zero, that is when $U_3z/y=0$, U_3 changes sign while $J_1(z)$ does not; also for $U_3=0$,

$$J_1(z)=\frac{z}{y}\,U_1,$$

and because of $U_2=0$, $\partial U_1/\partial z=0$, so that U_1 is a maximum or a minimum. At these points therefore $\partial^2 M^2/\partial z^2$ changes sign, and hence they also separate regions of the curve $U_2/y^2=0$ which give maxima from those which give minima, when the process described above of using the diagram is carried out.

The first three successive differential coefficients of M^2 all vanish when $z=0$, and $y=4m\pi$, that is at the double points,

and as there, as the reader may verify,

$$\frac{\partial^4 M^2}{\partial z^4}=\frac{3}{2}\frac{1}{m^2\pi^2},$$

the double points are places of minimum (zero) value of M^2.

The regions of the curves $U_2/y^2=0$ can now, starting from the double points, be easily identified as regions which give maxima or minima when the diagram is used in the manner described above. To mark regions which give minima they are ruled heavy in the diagram; the other regions, which give maxima, are ruled light.

Thus the first regions from the double points to a maximum or minimum of the curve, or to a point of crossing of $J_1(z)=0$, whichever comes first, are ruled heavy, then the region from that point to the next point at which U_2 changes sign is ruled light, and so on.

The lower regions of the curves $J_1(z)=0$, from the axis of z to the points of meeting with $U_2/y^2=0$, are ruled heavy; the next regions, from the first points of crossing to the second, light, and so on alternately. Thus the whole diagram is filled in.

As we have seen

$$U_2=V_0-\cos\left(\tfrac{1}{2}y+\frac{z^2}{2y}\right)=J_0(z)-\left(\frac{z}{y}\right)^2 J_2(z)+\ldots-\cos\left(\tfrac{1}{2}y+\frac{z^2}{2y}\right).$$

Hence as y increases in comparison with z, the equation

$$U_2=J_0(z)-\cos\tfrac{1}{2}y$$

more and more nearly holds. The reader may verify that the curve $J_0(z)-\cos\frac{1}{2}y=0$ meets the axis of y at the same points as the exact curve, and has there the same double tangents.

On the other hand, if y be made smaller in comparison with z, then by 3 we have more and more nearly

$$\frac{2}{y}U_1=\frac{2}{z}J_1(z),\quad \frac{2}{y}U_2=2\frac{y}{z^2}J_2(z),$$

so that the branches of $U_2/y^2=0$ approach more and more nearly to the lines $J_2(z)=0$. Thus we verify the statement made at p. 187 above.

The value of M^2, namely

$$\left(\frac{2}{y}\right)^2(U_1^2+U_2^2),$$

with increasing z and stationary y, that is with increasing obliquity of the rays, approaches zero. Hence at a great distance from the geometrical image of the orifice the illumination is practically zero.

Consider a line drawn in the diagram to fulfil the equation $y = cz$. A line making the same angle with the axis of y would have the equation $y = \frac{1}{c} z$. Let us consider the intensities for points on these two lines.

Since $y/z = c$ for the first line, then for any point on that line

$$\left.\begin{aligned} U_1 &= cJ_1 - c^3J_3 + \dots \\ &= \sin\left\{\tfrac{1}{2}z\left(c + \frac{1}{c}\right)\right\} - \left(\frac{1}{c}J_1 - \frac{1}{c^3}J_3 + \frac{1}{c^5}J_5 - \dots\right) \\ U_2 &= c^2J_2 - c^4J_4 + \dots \\ &= -\cos\left\{\tfrac{1}{2}z\left(c + \frac{1}{c}\right)\right\} + J_0 - \frac{1}{c^2}J_2 + \frac{1}{c^4}J_4 - \dots \end{aligned}\right\}. \qquad 47$$

For the other line we have, accenting the functions for distinction,

$$\left.\begin{aligned} U_1' &= \frac{1}{c}J_1 - \frac{1}{c^3}J_3 + \dots \\ &= \sin\left\{\tfrac{1}{2}z\left(c + \frac{1}{c}\right)\right\} - (cJ_1 - c^3J_3 + c^5J_5 - \dots), \\ U_2' &= \frac{1}{c^2}J_2 - \frac{1}{c^4}J_4 + \dots \\ &= -\cos\left\{\tfrac{1}{2}z\left(c + \frac{1}{c}\right)\right\} + J_0 - c^2J_2 + c^4J_4 - \dots \end{aligned}\right\}. \qquad 47'$$

Therefore

$$\left.\begin{aligned} U_1 + U_1' &= \sin\left\{\tfrac{1}{2}z\left(c + \frac{1}{c}\right)\right\}, \\ U_2 + U_2' &= J_0(z) - \cos\left\{\tfrac{1}{2}z\left(c + \frac{1}{c}\right)\right\} \end{aligned}\right\}. \qquad 48$$

Now if the radius of the geometrical shadow be ζ_0, then

$$\zeta_0 = (a + b)\, r/a,$$

and

$$\frac{y}{z} = \frac{\zeta_0}{\zeta} = c.$$

If ζ' be the distance of a point of the illuminated area upon the other line $y = z/c$ we have evidently

$$\zeta\zeta' = \zeta_0^2.$$

As special cases of these lines we have $z = 0$, or the axis of y, $y = 0$ or the axis of z, and $y = z$. The last is dotted in the diagram, and by the result just stated corresponds to the edge of the geometrical shadow.

The intensities for points along the first line are the intensities at the axis of symmetry for different radii of the orifice, or with constant radius for different values of b, the distance of the screen from the orifice. Those for points along the second line are the intensities for the case of Fraunhofer, already fully considered.

In the first case we have by 20, since $z = 0$,

$$U_1 = \sin \tfrac{1}{2}y, \quad U_2 = 2\sin^2 \tfrac{1}{4}y$$

so that

$$M^2 = \left(\frac{2}{y}\right)^2 (U_1^2 + U_2^2)$$

$$= \left(\frac{\sin \frac{1}{4}y}{\frac{1}{4}y}\right)^2. \qquad 49$$

This is the expression for the intensity at a point of a screen, produced by diffraction through a narrow slit, $\frac{1}{4}y$ in that case denoting $2\pi a\xi/\lambda f$, where a is the half breadth of the slit, f the distance of the illuminated point from the slit, and ξ the distance of the point from the geometrical image of the slit on the screen. Thus Tables, which have been prepared for the calculation of the brightness in the latter case, are available also for calculating the brightness at the centre of the geometrical image of the circular orifice.

The intensity is zero when $\frac{1}{4}y = m\pi$ (m being any whole number, zero excluded), that is when the difference of path between the extreme and central rays is a whole number of wave-lengths. It is a maximum when $\tan \frac{1}{4}y = \frac{1}{4}y$,

or $$\tan\left(\frac{\pi}{\lambda}\frac{a+b}{2ab}r^2\right) = \frac{\pi}{\lambda}\frac{a+b}{2ab}r^2.$$

Some values are given in the following Table.

$$z=0.$$

$\frac{1}{4}y$	$M^2=\left(\frac{\sin\frac{1}{4}y}{\frac{1}{4}y}\right)^2$
0·000000	1·000000
4·493409	0·047190
7·725252	0·016480
10·904120	0·008340
14·066194	0·005029
17·220753	0·003361
20·371302	0·002404
23·519446	0·001805
26·666054	0·001404

As y increases these values of $\frac{1}{4}y$ are given approximately by the equations

$$\tfrac{1}{4}y=\frac{2m+1}{2}\pi,$$

or

$$\left(\frac{1}{a}+\frac{1}{b}\right)\frac{r^2}{2}=\frac{2m+1}{2}\lambda,$$

that is the difference of path between the extreme and central rays is an odd number of half wave-lengths. The maximum intensity is then $16/y^2$, that is (as will be shown presently) four times the intensity at the screen due to the uninterrupted wave.

For the line $y=z$ in the diagram which corresponds to the edge of the geometrical shadow, we have by 13

$$U_1=\frac{1}{2}\sin z,$$

$$U_2=\frac{1}{2}(J_0(z)-\cos z),$$

so that

$$M^2=\frac{1}{z^2}\{\sin^2 z+(J_0(z)-\cos z)^2\}. \qquad 50$$

Clearly M^2 cannot vanish unless $\sin z$ and $J_0(z)-\cos z$ vanish separately, that is unless $J_0(z)=1$, which is impossible unless $z=0$.

It remains finally to find the illumination at a point on the screen when the orifice is replaced by an opaque disk, all the rest

of the wave being allowed to pass unimpeded. Going back to the original expressions, obtained at p. 167 above for the intensity, we see that for the total effect of the uninterrupted wave we have by 1 and 2

$$\left.\begin{aligned} C_\infty &= 2\pi\int_0^\infty J_0(l\rho)\cos(\tfrac{1}{2}k\rho^2)\,\rho d\rho = \pi\frac{2}{k}\sin\frac{l^2}{2k} \\ S_\infty &= 2\pi\int_0^\infty J_0(l\rho)\sin(\tfrac{1}{2}k\rho^2)\,\rho d\rho = \pi\frac{2}{k}\cos\frac{l^2}{2k} \end{aligned}\right\}. \qquad 51$$

Thus we get

$$M^2 = \pi^2\left(\frac{2}{k}\right)^2 = \pi r^2\left(\frac{2}{kr}\right)^2 = \left(\frac{2}{y}\right)^2, \qquad 52$$

if, as at p. 170 above, πr^2 be taken as unity. This is as it ought to be, as it leads to the expression $1/(a+b)^2$ for the intensity at the point in which the axis meets the screen. We thus verify the statement, made on the last page, that the maximum illumination at the centre of the geometrical image of the orifice is four times that due to the uninterrupted wave.

It might be objected that the original expressions obtained, which are here extended to the whole wave-front, had reference only to a small part of the wave-front, namely that filling the orifice. It is to be observed however that the effects of those elements of the wave-front, which lie at a distance from the axis, are very small compared with those of the elements near the axis, and so the integrals can be extended as above without error.

To find the illumination with the opaque disk we have simply to subtract from the values of C_∞, S_∞ the values of C_r, S_r, given on p. 169 for the orifice. Thus denoting the differences by C_1, S_1 we get

$$\left.\begin{aligned} C_1 &= C_\infty - C_r = -\frac{2}{y}(V_0\sin\tfrac{1}{2}y - V_1\cos\tfrac{1}{2}y) \\ S_1 &= S_\infty - S_r = \frac{2}{y}(V_0\cos\tfrac{1}{2}y + V_1\sin\tfrac{1}{2}y) \end{aligned}\right\}, \qquad 53$$

and

$$M_1^2 = \left(\frac{2}{y}\right)^2(V_0^2 + V_1^2), \qquad 54$$

or

$$M_1^2 = \left(\frac{2}{y}\right)^2\left\{1 + U_1^2 + U_2^2 - 2U_1\sin\tfrac{1}{2}\left(y + \frac{z^2}{y}\right) + 2U_2\cos\tfrac{1}{2}\left(y + \frac{z^2}{y}\right)\right\}. \qquad 55$$

Comparing these with the expressions on p. 170 for M^2 we see that they are the same except that now U_1 is replaced by V_1 and U_2 by $-V_0$.

If $z = 0$, that is if the point considered be at the centre of the geometrical shadow, $V_0 = 1$, $V_1 = 0$, and

$$M_1^2 = \left(\frac{2}{y}\right)^2, \tag{56}$$

that is the brightness there is always the same, exactly, as if the opaque disk did not exist. This is the well-known theoretical result first pointed out by Poisson, and since verified by experiment.

For any given values of y and z M_1^2 is easily calculated from those of U_1, U_2 by the equations 9

$$V_0 = \cos \tfrac{1}{2}\left(y + \frac{z^2}{y}\right) + U_2,$$

$$V_1 = \sin \tfrac{1}{2}\left(y + \frac{z^2}{y}\right) - U_1.$$

A valuable set of numerical Tables of M_1^2 all fully illustrated by curves will be found in Lommel's memoir.

When z is continually increased in value the equations

$$V_0 = \cos \tfrac{1}{2}\left(y + \frac{z^2}{y}\right),$$

$$V_1 = \sin \tfrac{1}{2}\left(y + \frac{z^2}{y}\right)$$

more and more nearly hold, since, by 11, U_2 and U_1 continually approach zero. The value of M_1^2 thus becomes $4/y^2$ at a great distance from the shadow, as in the uninterrupted wave.

As before we can find the conditions for a maximum or minimum. Differentiating, and reducing by 18 and 14, we obtain

$$\frac{\partial M_1^2}{\partial z} = -2\left(\frac{2}{y}\right)^2 J_1(z)\, V_0. \tag{57}$$

The maxima and minima have place therefore when

$$J_1(z) = 0, \text{ or } V_0 = 0.$$

The roots of these equations are the values of z which satisfy

$$\frac{\partial J_0(z)}{\partial z} = 0, \quad \frac{y}{z}\frac{\partial V_1}{\partial z} = 0,$$

and are, therefore, values of z which make $J_0(z)$ and V_1 a maximum or minimum. The roots of $J_1(z)=0$ are given at the end of this book; those of $V_0=0$ can be found by a formula of interpolation similar to, and obtained in the same way as, 43 above.

The tangent of the inclination of the curves $V_0=0$ to the axis of z is given according to 24 by

$$\frac{dy}{dz}=\frac{2\frac{z}{y}V_{-1}}{V_1+\left(\frac{z}{y}\right)^2 V_{-1}}. \qquad 58$$

By using in this the values

$$V_1=\frac{z}{y}J_1-\left(\frac{z}{y}\right)^3 J_3+\dots,$$

$$\left(\frac{z}{y}\right)^2 V_{-1}=-\frac{z}{y}J_1-\left(\frac{z}{y}\right)^3 J_1+\left(\frac{z}{y}\right)^5 J_3-\dots,$$

we see that if $y=\infty$, $\frac{dy}{dz}=\infty$, that is the curves are for great values of y parallel to the axis of y. Also since

$$V_0=J_0-\left(\frac{z}{y}\right)^2 J_2+\dots,$$

the asymptotes of these curves are the lines

$$J_0(z)=0,$$

drawn parallel to the axis of y. A table of the roots of this equation is given at the end of this book, and as has been seen above (p. 46) their large values are given approximately by the formula

$$(m+\tfrac{3}{4})\pi.$$

Writing now

$$V_0=\cos\tfrac{1}{2}\left(y+\frac{z^2}{y}\right)+U_2$$

$$=\cos\tfrac{1}{2}\left(y+\frac{z^2}{y}\right)+\left(\frac{y}{z}\right)^2 J_2-\left(\frac{y}{z}\right)^4 J_4+\dots=0,$$

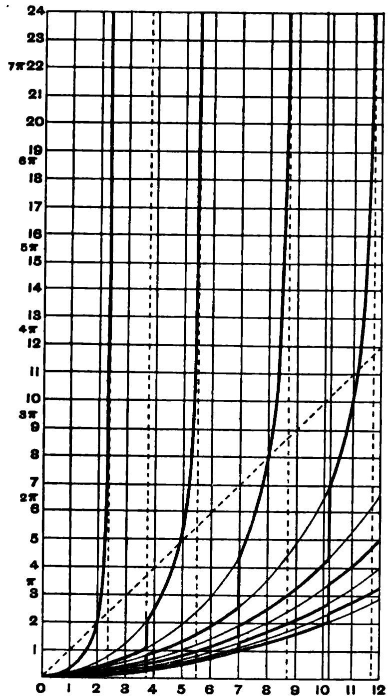
and making the values of y, z small, the terms after the first all disappear, and we are left with

$$\cos\tfrac{1}{2}\left(y+\frac{z^2}{y}\right)=0,$$

that is $$y^2+z^2=(2m+1)\pi y. \qquad 59$$

This equation represents a circle passing through the origin. We infer that the branches of the curve $V_0 = 0$ become near the origin arcs of circles all touching the axis of z at the origin.

The curves $V_0 = 0$ are shown in the adjoining diagram, and



give the maxima and minima of the intensity curve by the process already described for the diagram at p. 185.

In this case

$$\frac{\partial^2 M_1^2}{\partial z^2} = 2\left(\frac{2}{y}\right)^2 \left\{J_1^2 + \frac{z}{y} J_1 V_1 + \left(\frac{1}{z} J_1 - J_0\right) V_0\right\}, \qquad 60$$

so that points in which a line drawn parallel to the axis of z across the diagram cuts the lines $J_1(z) = 0$, $V_0 = 0$, correspond to maxima or minima according as the quantity on the right is negative or positive. Hence along the line $J_1(z) = 0$, the intensity is a maximum or a minimum according as $J_0 V_0$ is positive or negative. On the other hand, where $V_0 = 0$, the points correspond to maxima or minima according as

$$J_1^2(z) + \frac{z}{y} J_1(z) V_1 \quad \text{or} \quad -\frac{z}{y} J_1(z) V_{-1}$$

is negative or positive.

Where both $J_1(z)=0$ and $V_0=0$ the value of $\frac{\partial M_1^2}{\partial z^2}$ vanishes, but not so that of $\frac{\partial^3 M_1^2}{\partial z^3}$. Hence at such points the curves of intensity have points of inflexion, but there are no others.

As in the other case, points of inflexion of the intensity curve can only exist outside the shadow region of the diagram. For since $J_1(z)=0$, $V_{-1}=-V_1$, 58 becomes

$$\frac{dy}{dz}=-2\frac{\frac{z}{y}}{1-\left(\frac{z}{y}\right)^2},$$

which is positive if $y<z$, negative if $y>z$. But by the diagram $\frac{dy}{dz}$ is positive everywhere. Hence there can be no intersection of the line $J_1(z)=0$ with $V_0=0$, except when $y<z$. Thus the statement just made is proved.

Lastly, for the sake of comparing further the case of the disk with that of the orifice, let us contrast the intensity along a line $y=cz$ with that along the line $y=z/c$. Accenting the quantities for the second line, we can easily prove that

$$V_0^2+V_1^2+V_0'^2+V_1'^2=U_1^2+U_2^2+U_1'^2+U_2'^2+2J_0(z)\cos\tfrac{1}{2}z\left(c+\frac{1}{c}\right). \qquad 61$$

Now we have for the orifice

$$M^2=\frac{1}{c^2}\left(\frac{2}{z}\right)^2(U_1^2+U_2^2),$$

$$M'^2=c^2\left(\frac{2}{z}\right)^2(U_1'^2+U_2'^2),$$

and for the disk

$$M_1^2=\frac{1}{c^2}\left(\frac{2}{z}\right)^2(V_0+V_1^2),$$

$$M_1'^2=c^2\left(\frac{2}{z}\right)^2(V_0'^2+V_1'^2).$$

Thus by 61

$$c^2(M_1^2-M^2)+\frac{1}{c^2}(M_1'^2-M'^2)=\frac{8}{z^2}J_0(z)\cos\tfrac{1}{2}z\left(c+\frac{1}{c}\right). \qquad 62$$

The shadow region is that for which $y > z$, and is bounded therefore by the line $y = z$. On this line $M = M'$, $M_1 = M_1'$ and $c = 1$, so that 62 becomes

$$2(M_1^2 - M^2)_{y=z} = \frac{8}{z^2} J_0(z) \cos z.$$

It is clear from the diagram that as y increases the number of dark rings which fall within the shadow also increases.

The reader must refer for further information on these cases of diffraction to Lommel's paper, which contains, as we have indicated, a wealth of numerical and graphical results of great value. The discussion given above is in great part an account of this memoir, with deviations here and there from the original in the proofs of various theorems, and making use of the properties of Bessel functions established in the earlier chapters of this book.

The same volume of the *Abhandlungen der Königl. Bayer. Akademie der Wissenschaften* contains another most elaborate memoir by Lommel on the diffraction of a screen bounded by straight edges, in which the analysis is in many respects similar to that used in the first paper, and given above. We can only here find space for some particular applications therein made of Bessel Functions to the calculation of Fresnel's integrals, and a few other results.

From the result obtained above for Fraunhofer's interference phenomena, namely that the intensity of illumination is proportional to $\frac{4}{z^2} J_1^2(z)$, the source of light being a point, we can find the intensity at any point of the screen when the source is a uniform straight line arrangement of independent point-sources.

Let the circular orifice be the opening of the object-glass of the telescope which in Fraunhofer's experiments is supposed focused on the source of light. If the source is at a great distance from the telescope we may suppose with sufficient accuracy that the plane of the orifice is at right angles to the ray coming from any point of the linear source.

Let rectangular axes of ξ, η be drawn on the screen, and let the line of sources be parallel to the axis of η and in the plane $\xi = 0$. A little consideration shows that the illumination at any

point of the screen must depend upon ξ and (constant factors omitted) be represented by

$$\int_0^\infty \frac{J_1^2(z)}{z^2}\,d\eta.$$

But if r be the radius of the object-glass, and ζ the distance of the point considered from the axis of the telescope,

$$z = \frac{2\pi r}{b\lambda}\zeta = \mu\zeta, \text{ say,}$$

and $$\eta^2 = \zeta^2 - \xi^2 = \frac{z^2}{\mu^2} - \xi^2.$$

Hence $$d\eta = \frac{z}{\mu^2}\frac{dz}{\eta} = \frac{z}{\mu}\frac{dz}{\sqrt{z^2 - v^2}},$$

if $v^2 = \mu^2\xi^2$. The integral is therefore

$$\frac{1}{\mu}\int_v^\infty \frac{J_1^2(z)\,dz}{z\sqrt{z^2 - v^2}}.$$

This integral may be transformed in various ways into a form suitable for numerical calculation. The process here adopted depends on the properties of Bessel functions, and is due to Dr H. Struve*. Another method of obtaining the same result will be found in Lord Rayleigh's *Wave Theory of Light*†.

Before, however, we can give Struve's analysis we have to prove three lemmas on which his process depends.

The first is a theorem of Neumann's and is expressed by the equation

$$J_n^2(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} J_{2n}(2z\sin\alpha)\,d\alpha. \qquad 63$$

By 72, p. 28 above, we have

$$J_0\left(2c\sin\frac{\alpha}{2}\right) = J_0^2(c) + 2J_1^2(c)\cos\alpha + 2J_2^2(c)\cos 2\alpha + \dots.$$

But $$J_0\left(2c\sin\frac{\alpha}{2}\right) = \frac{1}{\pi}\int_0^\pi \cos\left(2c\sin\frac{\alpha}{2}\sin\phi\right)d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \{J_0(2c\sin\phi) + 2J_2(2c\sin\phi)\cos\alpha + 2J_4(2c\sin\phi)\cos 2\alpha + \dots\}\,d\phi$$

* *Wied. Ann.* 16, (1882), p. 1008. † *Encyc. Brit.* 9th Ed., p. 433.

by 40, p. 18 above. Identifying terms in the two equations we obtain

$$J_n^2(c) = \frac{1}{\pi}\int_0^\pi J_{2n}(2c\sin\phi)\,d\phi,$$

or

$$J_n^2(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{2n}(2z\sin\alpha)\,d\alpha,$$

if we write z for c and α for ϕ. Thus the first lemma is established.

The second lemma is the equation

$$J_0(vx) = \frac{2}{\pi}\int_v^\infty \frac{\sin(xz)}{\sqrt{z^2 - v^2}}\,dz, \qquad 64$$

$v \geqq 0$.

If P_n denote the zonal harmonic of the nth order, then it is a theorem of Dirichlet's that

$$\frac{\pi}{2}P_n(\cos\theta) = \int_0^\theta \frac{\cos\frac{1}{2}\phi\cos n\phi}{\sqrt{2(\cos\phi - \cos\theta)}}\,d\phi + \int_\theta^\pi \frac{\sin\frac{1}{2}\phi\cos n\phi}{\sqrt{2(\cos\theta - \cos\phi)}}\,d\phi$$

$$= -\int_0^\theta \frac{\sin\frac{1}{2}\phi\sin n\phi}{\sqrt{2(\cos\phi - \cos\theta)}}\,d\phi + \int_\theta^\pi \frac{\cos\frac{1}{2}\phi\sin n\phi}{\sqrt{2(\cos\theta - \cos\phi)}}\,d\phi.$$

Subtracting the second expression on the right from the first we obtain

$$\int_0^\theta \frac{\cos(n-\frac{1}{2})\phi}{\sqrt{2(\cos\phi - \cos\theta)}}\,d\phi = \int_\theta^\pi \frac{\sin(n-\frac{1}{2})\phi}{\sqrt{2(\cos\theta - \cos\phi)}}\,d\phi.$$

For θ and ϕ put θ/n, ϕ/n, and let n be made very great; the last equation becomes

$$\int_0^\theta \frac{\cos\phi\,d\phi}{\sqrt{\theta^2 - \phi^2}} = \int_\theta^\infty \frac{\sin\phi\,d\phi}{\sqrt{\phi^2 - \theta^2}}.$$

The quantity on the left is $\frac{1}{2}\pi J_0(\theta)$, (see p. 32 above). Hence

$$\frac{\pi}{2}J_0(\theta) = \int_\theta^\infty \frac{\sin\phi\,d\phi}{\sqrt{\phi^2 - \theta^2}}. \qquad 65$$

Write xz for ϕ, and vx for θ, xdz for $d\phi$ and 65 becomes

$$\frac{\pi}{2}J_0(vx) = \int_v^\infty \frac{\sin(xz)\,dz}{\sqrt{z^2 - v^2}},$$

which is the second lemma stated above.

The third lemma is expressed by the equation

$$\int_0^{\frac{1}{2}\pi} J_0(z\sin\alpha)\sin\alpha d\alpha = \frac{\sin z}{z}. \qquad 66$$

Using the general definition (p. 12 above) of a Bessel function of integral order we get

$$\int_0^{\frac{1}{2}\pi} J_0(z\sin\alpha)\sin\alpha d\alpha = \sum_0^\infty \frac{(-)^s}{(\Pi s)^2}\left(\frac{z}{2}\right)^{2s}\int_0^{\frac{1}{2}\pi}\sin^{2s+1}\alpha d\alpha$$

$$= \Sigma \frac{(-)^s}{(\Pi s)^2}\left(\frac{z}{2}\right)^{2s}\frac{2^{2s}(\Pi s)^2}{\Pi(2s+1)}$$

$$= \Sigma \frac{(-)^s z^{2s}}{\Pi(2s+1)} = \frac{\sin z}{z},$$

which was to be proved.

Returning now to the integral

$$\int_v^\infty \frac{J_1^2(z)\,dz}{z\sqrt{z^2-v^2}}$$

let us denote it by Z. We have by the first lemma

$$Z = \frac{2}{\pi}\int_v^\infty \frac{dz}{z\sqrt{z^2-v^2}}\int_0^{\frac{1}{2}\pi} J_2(2z\sin\alpha)\,d\alpha.$$

But by 20, p. 13

$$J_2(2z\sin\alpha) = \frac{z\sin\alpha}{2}\{J_1(2z\sin\alpha) + J_3(2z\sin\alpha)\},$$

and by 45, p. 18

$$J_1(2z\sin\alpha) = \frac{1}{\pi}\int_0^\pi \sin(2z\sin\alpha\sin\beta)\sin\beta d\beta,$$

$$J_3(2z\sin\alpha) = \frac{1}{\pi}\int_0^\pi \sin(2z\sin\alpha\sin\beta)\sin 3\beta d\beta,$$

so that

$$Z = \frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}\sin\alpha d\alpha\int_0^\pi(\sin\beta+\sin 3\beta)\,d\beta\int_v^\infty \frac{\sin(2z\sin\alpha\sin\beta)\,dz}{\sqrt{z^2-v^2}}.$$

But if we put $2\sin\alpha\sin\beta = x$ we get by the second lemma

$$\int_v^\infty \frac{\sin(2z\sin\alpha\sin\beta)\,dz}{\sqrt{z^2-v^2}} = \frac{\pi}{2}J_0(vx).$$

Hence

$$Z = \frac{1}{2\pi}\int_0^\pi(\sin\beta+\sin 3\beta)\,d\beta\int_0^{\frac{1}{2}\pi}J_0(2v\sin\alpha\sin\beta)\sin\alpha d\alpha,$$

which by the third lemma becomes

$$Z = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} (\sin\beta + \sin 3\beta)\frac{\sin(2v\sin\beta)}{2v\sin\beta}\,d\beta$$

$$= \frac{2}{\pi v}\int_0^{\frac{1}{2}\pi} \sin(2v\sin\beta)\cos^2\beta\,d\beta. \qquad 67$$

Let now $H_0(z)$ be a function defined by the equation

$$H_0(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\,d\theta. \qquad 68$$

Expanding $\sin(z\sin\theta)$ and integrating we obtain

$$H_0(z) = \frac{2}{\pi}\left\{z - \frac{z^3}{1^2.3^2} + \frac{z^5}{1^2.3^2.5^2} - \dots\right\}. \qquad 69$$

Now let $H_1(z)$ be another function defined by

$$H_1(z) = \int_0^z H_0(z)\,z\,dz,$$

then by the series in 69

$$H_1(z) = \frac{2}{\pi}\left\{\frac{z^3}{1^2.3} - \frac{z^5}{1^2.3^2.5} + \frac{z^7}{1^2.3^2.5^2.7} - \dots\right\}. \qquad 70$$

We shall now prove that

$$H_1(z) = \frac{2z^2}{\pi}\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\cos^2\theta\,d\theta. \qquad 71$$

It can be verified by differentiating that

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{d}{dz}\right)H_0(z) = \frac{2}{\pi z} - H_0(z). \qquad 72$$

Multiplying by $z\,dz$ and integrating we find

$$H_1(z) = \frac{2z}{\pi} - z\frac{dH_0(z)}{dz}. \qquad 73$$

Now by 68

$$z\frac{dH_0(z)}{dz} = \frac{2z}{\pi}\int_0^{\frac{1}{2}\pi} \cos(z\sin\theta)\sin\theta\,d\theta.$$

Hence

$$H_1(z) = \frac{2z}{\pi}\left\{1 - \int_0^{\frac{1}{2}\pi} \cos(z\sin\theta)\sin\theta\,d\theta\right\}$$

$$= \frac{2z}{\pi}\int_0^{\frac{1}{2}\pi} \{1 - \cos(z\sin\theta)\}\sin\theta\,d\theta$$

$$= \frac{4z}{\pi}\int_0^{\frac{1}{2}\pi} \sin^2(\tfrac{1}{2}z\sin\theta)\sin\theta\,d\theta. \qquad 74$$

It may be noted that every element of this integral is positive. It is clear from the form of $H_1(z)$ given in the first of the three equations just written that $H_1(z)$ approximates when z is large to $2z/\pi$.

Integrating 74 by parts we obtain

$$H_1(z) = \frac{2z^2}{\pi}\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\cos^2\theta\, d\theta,$$

since the integrated term vanishes at both limits.

If we write $2v$ for z and β for θ the equation becomes

$$H_1(2v) = 2\,\frac{(2v)^2}{\pi}\int_0^{\frac{1}{2}\pi} \sin(2v\sin\beta)\cos^2\beta\, d\beta. \qquad 75$$

Hence

$$\frac{H_1(2v)}{4v^3} = \frac{2}{\pi v}\int_0^{\frac{1}{2}\pi} \sin(2v\sin\beta)\cos^2\beta\, d\beta = Z,$$

by 67.

It is to be observed that the functions here denoted by $H_0(z)$, $H_1(z)$ are the same as Lord Rayleigh's $K(z)$, $K_1(z)$ discussed in the *Theory of Sound*, § 302, to which the reader is referred for further details. The function $H_1(z)$ differs however from the function $H_1(z)$ used by Struve. If we denote the latter by $\mathfrak{H}_1(z)$, the relation between the two functions is

$$H_1(z) = z\mathfrak{H}_1(z).$$

The value of $H_1(z)$ can be calculated when z is not too great by the series in 70, but when z is large this series is not convenient. We must then have recourse to a semiconvergent series, similar to that established in Chap. IV. above for the Bessel functions. The series can be found easily by the method of Lipschitz, already used in Chap. VII. The following is a brief outline of the process*.

By the definitions of the functions we have

$$J_0(z) - iH_0(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{-iz\sin\theta}d\theta$$

$$= \frac{2}{\pi}\int_0^1 \frac{e^{-ivz}}{\sqrt{1-v^2}}\,dv.$$

* See *Theory of Sound*, § 302.

Now take the integral

$$\int \frac{e^{-zw}\,dw}{\sqrt{1+w^2}}$$

(in which $w = u + iv$) round the rectangle, the angular points of which are 0, h, $h+i$, i where h is real and positive. This integral is zero, and if $h = \infty$ it gives after some reduction

$$\int_0^1 \frac{e^{-izv}}{\sqrt{1-v^2}}\,dv = -\frac{i}{z}\int_0^\infty e^{-\beta}\left(1+\frac{\beta^2}{z^2}\right)^{-\frac{1}{2}} d\beta$$
$$+\frac{e^{-i(z-\frac{1}{4}\pi)}}{\sqrt{2z}}\int_0^\infty e^{-\beta}\beta^{-\frac{1}{2}}\left(1-\frac{i\beta}{2z}\right)^{-\frac{1}{2}} d\beta.$$

Expanding the binomials and integrating, making use of the theorem

$$\int_0^\infty e^{-\beta}\beta^{q-\frac{1}{2}}\,d\beta = \Pi\,(q-\tfrac{1}{2}),$$

and equating the real part of the result to $\frac{1}{2}\pi J_0(z)$ and the imaginary part to $-\frac{1}{2}i\pi H_0(z)$, we get the expansions required, namely $J_0(z)$ as in Chap. IV. and

$$H_0(z) = \frac{2}{\pi}(z^{-1} - z^{-3} + 1^2.3^2z^{-5} - 1^2.3^2.5^2z^{-7} + \ldots)$$
$$+\sqrt{\frac{2}{\pi z}}\,\{P\sin(z-\tfrac{1}{4}\pi) - Q\cos(z-\tfrac{1}{4}\pi)\}, \qquad 76$$

where P and Q have the values stated on p. 48 above (z being written for x).

From this the value of $H_1(z)$ is at once found by the relation 73 and is

$$H_1(z) = \frac{2}{\pi}(z + z^{-1} - 3z^{-3} + 1^2.3^2.5z^{-5} - \ldots)$$
$$-\sqrt{\frac{2z}{\pi}}\cos(z-\tfrac{1}{4}\pi)\left\{1 - \frac{(1^2-4)(3^2-4)}{1.2.(8z)^2}\right.$$
$$\left.+\frac{(1^2-4)(3^2-4)(5^2-4)(7^2-4)}{1.2.3.4.(8z)^4} - \ldots\right\}$$
$$-\sqrt{\frac{2z}{\pi}}\sin(z-\tfrac{1}{4}\pi)\left\{\frac{1^2-4}{1.8z}\right.$$
$$\left.-\frac{(1^2-4)(3^2-4)(5^2-4)}{1.2.3.(8z)^3} + \ldots\right\}. \qquad 77$$

It is to be noticed that $H_1(2v)$ is nowhere zero, and that $H_1(2v)/v^3$ has maxima and minima values at points satisfying the equation

$$\frac{d}{dv}\frac{H_1(2v)}{v^3} = \frac{4v^2 H_0(2v) - 3H_1(2v)}{v^4} = 0. \qquad 78$$

The corresponding values of v are therefore the roots of

$$4v^2 H_0(2v) - 3H_1(2v) = 0.$$

Now let there be two parallel and equally luminous line-sources, whose images in the focal plane are at a distance apart $v/\mu = \pi/\mu$, say. It is of great importance to compare the intensity at the image of either line with the intensity halfway between them. In this way can be determined the minimum distance apart at which the luminous lines may be placed and still be separated by the telescope. We shall take the image of one as corresponding to $v = 0$, and that of the other as corresponding to $v = \pi$. Thus the intensity at any distance corresponding to v is proportional to $\dfrac{H_1(2v)}{4v^3}$. Putting

$$L(v) = \frac{\pi}{2}\frac{H_1(2v)}{(2v)^3},$$

we have by 70

$$L(v) = \frac{1}{1^2 . 3} - \frac{2^2 v^2}{1^2 . 3^2 . 5} + \frac{2^4 v^4}{1^2 . 3^2 . 5^2 . 7} - \dots .$$

The ratio of the intensity of illumination midway between the two lines to that at either is therefore

$$\frac{2L(\frac{1}{2}\pi)}{L(0) + L(\pi)}.$$

This has been calculated by Lord Rayleigh (to whom this comparison is due) with the following results

$$L(0) = \cdot 3333, \quad L(\pi) = \cdot 0164, \quad L(\tfrac{1}{2}\pi) = \cdot 1671,$$

so that

$$\frac{2L(\frac{1}{2}\pi)}{L(0) + L(\pi)} = \cdot 955. \qquad 79$$

The intensity is therefore, for the distance stated, only about $4\frac{1}{2}$ per cent. less than at the image of either line.

Now

$$v = \frac{2\pi r}{\lambda b}\xi = \pi,$$

which gives

$$\frac{\xi}{b} = \frac{\lambda}{2r}.$$

Since b is the focal length of the object-glass, the two lines are, by this result, at an angular distance apart equal to that subtended by the wave-length of light at a distance equal to the diameter of the object-glass. Two lines unless at a greater angular distance could therefore hardly be separated.

This result shows that the resolving (or as it is sometimes called the space-penetrating) power of a telescope is directly proportional to the diameter of the object-glass.

By multiplying

$$\frac{2}{\pi}\frac{H_1(2v)}{(2v)^3}$$

by $\mu d\xi$, that is by dv, and integrating from $\xi = -\infty$ to $\xi = +\infty$ we get an expression which, to a constant factor, represents the whole illumination received by the screen from a single luminous point the image of which is at the centre of the focal plane. Or, by the mode in which $H_1(2v)/v^3$ was obtained, it plainly may be regarded as the illumination received by the latter point from an infinite uniformly illuminated area in front of the object-glass.

If the integral is taken from ξ to $+\infty$ it will represent on the same scale, the illumination received by the same point from an area bounded by the straight line parallel to η corresponding to the constant value of ξ. The point will be at a distance ξ from the edge of the geometrical shadow, and will be inside or outside the shadow according as ξ is positive or negative.

We have by 71

$$\int_0^\infty \frac{H_1(2v)}{(2v)^3}dv = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\cos^2\beta d\beta\int_0^\infty \frac{\sin(2v\sin\beta)}{v}dv$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^2\beta d\beta = \frac{\pi}{8}.$$

Now

$$\int_v^\infty \frac{H_1(2v)}{(2v)^3}dv = \int_0^\infty \frac{H_1(2v)}{(2v)^3}dv - \int_0^v \frac{H_1(2v)}{(2v)^3}dv.$$

The second term on the right can be calculated by means of the ascending series 70. Hence we get

$$\int_v^\infty \frac{H_1(2v)}{(2v)^3}\,dv = \frac{\pi}{8} - \frac{2}{\pi}\left\{\frac{v}{1^2 \cdot 3} - \frac{2^2 v^3}{1^2 \cdot 3^2 \cdot 3 \cdot 5} + \frac{2^4 v^5}{1^2 \cdot 3^2 \cdot 5^2 \cdot 5 \cdot 7} - \dots\right\}. \qquad 80$$

This multiplied by $4/\pi$ is the expression given by Struve for the intensity produced by a uniform plane source, the image of which extends from v to $+\infty$. For the sake of agreement with Struve's result we write when v is positive

$$I(+v) = \tfrac{1}{2} - \frac{4}{\pi^2}\left\{\frac{2v}{1^2 \cdot 3} - \frac{(2v)^3}{1^2 \cdot 3^2 \cdot 3 \cdot 5} + \dots\right\}. \qquad 81$$

Hence if I be the illumination when the plane source extends from $-\infty$ to $+\infty$ we have

$$I(+v) + I(-v) = I = 1.$$

This states that the intensities at two points equally distant from the edge of the geometrical shadow, but on opposite sides of it, are together equal to the full intensity. The intensity at the edge of the shadow is therefore half the full intensity.

The reader may verify that when v is great the semi-convergent series gives approximately

$$I(v) = \frac{2}{\pi^2}\left(\frac{1}{v} + \frac{1}{12v^3}\right) - \frac{1}{2\pi^{\frac{3}{2}}}\frac{\cos(2v + \frac{1}{4}\pi)}{v^{\frac{3}{2}}}.$$

The following Table (abridged from Struve's paper) gives the intensity within the geometrical shadow at a distance $\xi = b\lambda v/2\pi r$ from the edge, and therefore enables the enlargement of the image produced by the diffraction of the object-glass to be estimated.

$$v = \frac{2\pi r}{b\lambda}\xi.$$

$$I(-v) = 1 - I(+v).$$

v	$I(+v)$	v	$I(+v)$	v	$I(+v)$
0·0	·5000	2	·1073	7	·0293
0·5	·3678	3	·0630	9	·0222
1·0	·2521	4	·0528	11	·0186
1·5	·1642	5	·0410	15	·0135

We shall now consider very briefly the theory of diffraction of light passing through a narrow slit bounded by parallel edges. We shall suppose that the diffraction may be taken as the same in every plane at right angles to the slit, so that the problem is one in only two dimensions. Let a then be the radius of a circular wave that has just reached the gap, and consider an element of the wave-front in the gap. Let also b be the distance of P from the pole so that its distance from the source is $a+b$, ds the length of the element of the wave, and δ the retardation of the secondary wave (that is the difference between the distances of P from the element and from the pole). The disturbance at P produced will be proportional to

$$\cos 2\pi\left(\frac{t}{T}-\frac{\delta}{\lambda}\right) ds.$$

If the distance of the element from the pole be s, and s be small in comparison with b, then it is very easy to show that

$$\delta = \frac{a+b}{2ab} s^2.$$

Writing as usual $\frac{1}{2}\pi v^2$ for $2\pi\delta/\lambda$, we get

$$\frac{2\pi\delta}{\lambda} = \tfrac{1}{2}\pi v^2 = \frac{\pi(a+b)s^2}{ab\lambda}.$$

The disturbance at P is therefore

$$\cos 2\pi\left(\frac{t}{T}-\frac{v^2}{4}\right) = \cos \tfrac{1}{2}\pi v^2 \cos 2\pi \frac{t}{T} + \sin \tfrac{1}{2}\pi v^2 \sin 2\pi \frac{t}{T}.$$

The intensity of illumination due to the element is therefore constant, being proportional to

$$\cos^2 \tfrac{1}{2}\pi v^2 + \sin^2 \tfrac{1}{2}\pi v^2,$$

where

$$v^2 = \frac{2(a+b)}{ab\lambda} s^2.$$

The whole intensity is thus proportional to

$$\left\{\int \cos \tfrac{1}{2}\pi v^2 . dv\right\}^2 + \left\{\int \sin \tfrac{1}{2}\pi v^2 . dv\right\}^2,$$

the integrals being taken over the whole arc of the wave at the slit.

The problem is thus reduced to quadratures, and it remains to evaluate the integrals. We shall write

$$C=\int_0^v \cos \tfrac{1}{2}\pi v^2 dv, \quad S=\int_0^v \sin \tfrac{1}{2}\pi v^2 dv.$$

C and S are known as Fresnel's integrals.

Various methods of calculating these integrals have been devised; but the simplest of all for purposes of numerical calculation is by means of Bessel functions, when Tables are available.

Let $\frac{1}{2}\pi v^2 = z$, then

$$C=\tfrac{1}{2}\int_0^z \sqrt{\frac{2}{\pi z}}\cos z dz = \tfrac{1}{2}\int_0^z J_{-\frac{1}{2}}(z)\, dz, \qquad 82$$

$$S=\tfrac{1}{2}\int_0^z \sqrt{\frac{2}{\pi z}}\sin z dz = \tfrac{1}{2}\int_0^z J_{\frac{1}{2}}(z)\, dz. \qquad 83$$

Let us now consider the Bessel function integrals on the right. Using the relation

$$J'_n(z) = \tfrac{1}{2}\left(J_{n-1}(z) - J_{n+1}(z)\right)$$

we have

$$\begin{aligned} J_{-\frac{1}{2}}(z) &= 2J'_{\frac{1}{2}}(z) + J_{\frac{3}{2}}(z) \\ &= 2J'_{\frac{1}{2}}(z) + 2J'_{\frac{5}{2}}(z) + \dots + 2J'_{\frac{4n+1}{2}}(z) + J_{\frac{4n+3}{2}}(z). \end{aligned}$$

Thus we obtain

$$\tfrac{1}{2}\int_0^z J_{-\frac{1}{2}}(z)\, dz = J_{\frac{1}{2}}(z) + J_{\frac{5}{2}}(z) + \dots + J_{\frac{4n+1}{2}}(z) + \tfrac{1}{2}\int_0^z J_{\frac{4n+3}{2}}(z)\, dz. \qquad 84$$

By taking $(4n+3)/2$ sufficiently great the integral on the right of 84 may be made as small as we please. Thus we get

$$C = \tfrac{1}{2}\int_0^z J_{-\frac{1}{2}}(z)\, dz = J_{\frac{1}{2}}(z) + J_{\frac{5}{2}}(z) + J_{\frac{9}{2}}(z) + \dots. \qquad 85$$

Similarly we find

$$S = \tfrac{1}{2}\int_0^z J_{\frac{1}{2}}(z)\, dz = J_{\frac{3}{2}}(z) + J_{\frac{7}{2}}(z) + J_{\frac{11}{2}}(z) + \dots. \qquad 86$$

These series are convergent, and give the numerical value of the integrals to any degree of accuracy from Tables of Bessel functions of order $(2n+1)/2$, by simple addition of the values of the

successive alternate functions for the given argument. The series are apparently due to Lommel, and are stated in the second memoir referred to above, p. 198. He gives also the series

$$C = \tfrac{1}{2}\int_0^z J_{-\frac{1}{2}}(z)\,dz = \sqrt{2}\,(P\cos\tfrac{1}{2}z + Q\sin\tfrac{1}{2}z) \qquad 87$$

$$S = \tfrac{1}{2}\int_0^z J_{\frac{1}{2}}(z)\,dz = \sqrt{2}\,(P\sin\tfrac{1}{2}z - Q\cos\tfrac{1}{2}z) \qquad 88$$

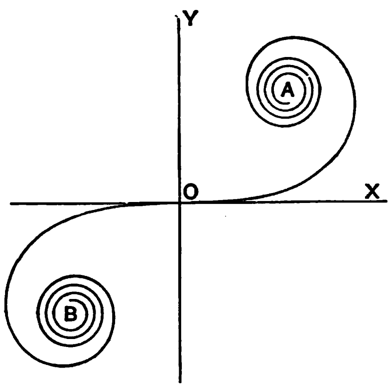
where

$$P = J_{\frac{1}{2}}(\tfrac{1}{2}z) - J_{\frac{5}{2}}(\tfrac{1}{2}z) + J_{\frac{9}{2}}(\tfrac{1}{2}z) - \dots,$$
$$Q = J_{\frac{3}{2}}(\tfrac{1}{2}z) - J_{\frac{7}{2}}(\tfrac{1}{2}z) + J_{\frac{11}{2}}(\tfrac{1}{2}z) - \dots.$$

The proof is left to the reader.

C and S were expressed long ago in series of ascending powers of v by Knochenhauer, and in terms of definite integrals by Gilbert. From the latter semi-convergent series suitable for use when v is large are obtainable by a process similar to that sketched at p. 203 above. It is not necessary however to pursue the matter here.

The very elegant construction shown in the diagram, which is known as Cornu's spiral, shows graphically how the value of $C^2 + S^2$ varies.



The abscissae of the curve are values of C and the ordinates values of S.

It can be shown that the distance along the curve from the origin to any point is the value of v for that point, that the inclination of the tangent to the axis of abscissae is $\frac{1}{2}\pi v^2$, and that the curvature there is πv.

As v varies from 0 to ∞ and from 0 to $-\infty$ the curve is wrapped more and more closely round the poles A and B.

The origin of the curve corresponds to the pole of the point considered, so that if v_1, v_2 correspond to the distances from the pole to the edges of the slit, we have only to mark the two points v_1, v_2 on the spiral and draw the chord. The square of the length

of this chord will represent the intensity of illumination at the point. The square of the length of the chord from the origin to any point v is the value of $C^2 + S^2$, that is of

$$\tfrac{1}{4}\left\{\int_0^z J_{-\frac{1}{2}}(z)\,dz\right\}^2 + \tfrac{1}{4}\left\{\int_0^z J_{\frac{1}{2}}(z)\,dz\right\}^2.$$

As v varies it will be seen that the value of this sum oscillates more and more rapidly while approaching more and more nearly to the value $\tfrac{1}{2}$.

CHAPTER XV.

MISCELLANEOUS APPLICATIONS.

IN this concluding chapter we propose to give a short account of some special applications of the Bessel functions which, although not so difficult as those already considered, appear too important or too interesting to be passed over entirely or simply placed in the collection of examples.

We will begin with the equation

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u, \qquad 1$$

which occurs in various physical problems, such as the small vibrations of a gas, or the variable flow of heat in a solid sphere.

Using polar coordinates, u is a function of t, r, θ, ϕ, such that

$$\frac{\partial u}{\partial t} = \frac{a^2}{r^2}\left\{r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right\}. \qquad 2$$

Assume, as a particular solution,

$$u = e^{-\kappa^2 a^2 t} v S_n,$$

where v is a function of r only, and S_n is a surface spherical harmonic of order n, so that

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\phi^2} + n(n+1)S_n = 0.$$

Then after substitution in (2), it appears that v must satisfy the equation

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + \left\{\kappa^2 - \frac{n(n+1)}{r^2}\right\}v = 0;$$

and now if we put

$$v = r^{-\frac{1}{2}} w,$$

we find that w satisfies the equation

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \left\{\kappa^2 - \frac{(n+\frac{1}{2})^2}{r^2}\right\} w = 0.$$

Hence

$$w = AJ_{n+\frac{1}{2}}(\kappa r) + BJ_{-n-\frac{1}{2}}(\kappa r),$$

and finally

$$u = e^{-\kappa^2 a^2 t}\left\{AJ_{n+\frac{1}{2}}(\kappa r) + BJ_{-n-\frac{1}{2}}(\kappa r)\right\}\frac{S_n}{\sqrt{r}}, \qquad 3$$

is a particular solution of the equation 1 or 2. In practice n is a whole number, and by a proper determination of the constants n, κ, A, B the function

$$U = \sum_{n,\,\kappa} e^{-\kappa^2 a^2 t}\left\{AJ_{n+\frac{1}{2}}(\kappa r) + BJ_{-n-\frac{1}{2}}(\kappa r)\right\}\frac{S_n}{\sqrt{r}}, \qquad 4$$

is adapted in the usual way to suit the particular conditions of the problem.

(See Riemann's *Partielle Differentialgleichungen*, pp. 176—189 and Rayleigh's *Theory of Sound*, chap. XVII.)

If in the above we suppose $n = 0$, the function S_n reduces to a constant, and

$$J_{n+\frac{1}{2}}(\kappa r) = \sqrt{\frac{2}{\pi\kappa r}}\sin\kappa r$$

(see p. 42); thus with a simplified notation, we have a solution

$$U = \sum_{\kappa} A_\kappa \frac{\sin\kappa r}{r} e^{-\kappa^2 a^2 t}, \qquad 5$$

which may be adapted to the following problem (Math. Tripos, 1886):

"A uniform homogeneous sphere of radius b is at uniform temperature V_0, and is surrounded by a spherical shell of the same substance of thickness b at temperature zero. The whole is left to cool in a medium at temperature zero. Prove that, after time t, the temperature at a point distant r from the centre is

$$V = V_0 \sum \frac{4}{\kappa}\frac{\sin\kappa b - \kappa b\cos\kappa b}{4\kappa b - \sin 4\kappa b}\frac{\sin\kappa r}{r} e^{-\kappa^2 a^2 t},$$

where the values of κ are given by the equation

$$\tan 2\kappa b = \frac{2\kappa b}{1-2hb},$$

h being the ratio of the surface conductivity to the internal conductivity."

Here the conditions to be satisfied are

$$V = V_0 \text{ from } r=0 \text{ to } r=b,$$
$$V=0 \quad \ldots \quad r=b \ldots r=2b,$$

when $t=0$; and

$$\frac{\partial V}{\partial r} + hV = 0,$$

when $r=2b$, for all values of t.

Now, assuming a solution of the form

$$V = \Sigma A_\kappa \frac{\sin \kappa r}{r} e^{-\kappa^2 a^2 t},$$

the last condition is satisfied if

$$\frac{\partial}{\partial r}\left(\frac{\sin \kappa r}{r}\right) + \frac{h \sin \kappa r}{r} = 0,$$

when $r=2b$: that is, if

$$\frac{\kappa \cos 2\kappa b}{2b} - \frac{\sin 2\kappa b}{4b^2} + \frac{h \sin 2\kappa b}{2b} = 0,$$

leading to

$$\tan 2\kappa b = \frac{2\kappa b}{1-2hb},$$

as above stated.

Proceeding as in Chap. VI. above, we infer that

$$A_\kappa \int_0^{2b} \sin^2 \kappa r dr = \int_0^{2b} Vr \sin \kappa r dr,$$

that is,

$$A_\kappa \left(b - \frac{\sin 4\kappa b}{4\kappa}\right) = V_0 \int_0^b r \sin \kappa r dr$$
$$= V_0 \left(\frac{\sin \kappa b}{\kappa^2} - \frac{b \cos \kappa b}{\kappa}\right),$$

and hence

$$A_\kappa = \frac{4}{\kappa} \cdot \frac{\sin \kappa b - \kappa b \cos \kappa b}{4\kappa b - \sin 4\kappa b} V_0,$$

which agrees with the result above given.

Returning to the solution given by equation 4 above, we may observe that when a^2 is real and positive, the solution is applicable to cases when there is a "damping" of the phenomenon considered, as in the problem just discussed. When there is a forced vibration imposed on the system, as when a spherical bell vibrates in air, we must take a^2 to be a pure imaginary $\pm i\alpha/\kappa$, so as to obtain a time-periodic solution. The period is then $2\pi/\kappa\alpha$. An illustration of this will be found at the end of the book.

We will now proceed to consider two problems suggested by the theory of elasticity.

The first is that of the stability of an isotropic circular cylinder of small cross-section held in a vertical position with its lower end clamped and upper end free.

It is a matter of common observation that a comparatively short piece of steel wire, such as a knitting-needle, is stable when placed vertically with its lower end clamped in a vice; whereas it would be impossible to keep vertical in the same way a very long piece of the same wire.

To find the greatest length consistent with stability, we consider the possibility of a position of equilibrium which only deviates *slightly* from the vertical.

Let w be the weight of the wire per unit of length, β its flexural rigidity. Then if x is the height of any point on the wire above the clamped end, and y its horizontal displacement from the vertical through that end, we obtain by taking moments for the part of the wire above (x, y)

$$\beta \frac{d^2y}{dx^2} = \int_x^l w\,(y' - y)\,dx',$$

l being the whole length of the wire.

Differentiate with respect to x; then

$$\begin{aligned}\beta \frac{d^3y}{dx^3} &= \int_x^l w\left(-\frac{dy}{dx}\right) dx' \\ &= -w\,(l-x)\frac{dy}{dx},\end{aligned}$$

or

$$\frac{d^2p}{dx^2} + \frac{w}{\beta}(l-x)\,p = 0,$$

if $\dfrac{dy}{dx} = p$.

Now put

$$l - x = r^{\frac{2}{3}}.$$

then

$$\frac{dp}{dx} = -\frac{3}{2} r^{\frac{1}{3}} \frac{dp}{dr},$$

$$\frac{d^2p}{dx^2} = \frac{9}{4}\left\{ r^{\frac{2}{3}} \frac{d^2p}{dr^2} + \frac{1}{3} r^{-\frac{1}{3}} \frac{dp}{dr} \right\},$$

and the transformed equation is

$$\frac{d^2p}{dr^2} + \frac{1}{3r}\frac{dp}{dr} + \frac{4w}{9\beta} p = 0;$$

and now, if we put

$$p = r^{\frac{1}{3}} z,$$

it will be found that

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} + \left(\frac{4w}{9\beta} - \frac{1}{9r^2}\right) z = 0.$$

Hence if

$$\kappa^2 = \frac{4w}{9\beta},$$

it follows that

$$z = AJ_{\frac{1}{3}}(\kappa r) + BJ_{-\frac{1}{3}}(\kappa r).$$

When $x = l$, that is, when $r = 0$, we must have

$$\frac{dp}{dx} = 0,$$

whence

$$r^{\frac{1}{3}} \frac{dp}{dr} = 0,$$

when $r = 0$; that is,

$$r^{\frac{1}{3}}\left\{ r^{\frac{1}{3}} \frac{dz}{dr} + \frac{1}{3} r^{-\frac{2}{3}} z \right\} = 0,$$

or

$$\frac{3r\dfrac{dz}{dr} + z}{3r^{\frac{1}{3}}} = 0.$$

Now the initial terms of $J_{\frac{1}{3}}(\kappa r)$ and $J_{-\frac{1}{3}}(\kappa r)$ are of the forms

$$J_{\frac{1}{3}}(\kappa r) = \alpha r^{\frac{1}{3}} + \beta r^{\frac{7}{3}} + \ldots,$$

$$J_{-\frac{1}{3}}(\kappa r) = \alpha' r^{-\frac{1}{3}} + \beta' r^{\frac{5}{3}} + \ldots,$$

and it is only the second of these that satisfies

$$r^{-\frac{1}{3}}\left(3r\frac{dz}{dr} + z\right) = 0,$$

when $r = 0$. Therefore $A = 0$.

Again, when $x = 0$, that is, when $r = l^{\frac{3}{2}}$, p, and therefore z, must be zero. Hence, in order that the assumed form of equilibrium may be possible,

$$J_{-\frac{1}{3}}(\kappa l^{\frac{3}{2}}) = 0.$$

The least value of l obtained from this equation gives the critical length of the wire when it first shows signs of instability in the vertical position; and if l is less than this, the vertical position will be stable.

It is found that the least root of

$$J_{-\frac{1}{3}}(x) = 0,$$

is approximately 1·88: so that the critical length is about

$$\left(\frac{1{\cdot}88}{\kappa}\right)^{\frac{2}{3}},$$

or

$$1{\cdot}996\sqrt[3]{\beta/w},$$

approximately.

To the degree of approximation adopted we may put

$$l = 2\sqrt[3]{\beta/w},$$

or in terms of β and W, the whole weight of the wire,

$$l = \sqrt{8\beta/W} = 2{\cdot}83\sqrt{\beta/W}.$$

Of the two formulae given the first is the proper one for determining the critical length for a given kind of wire; the second is convenient if we wish to know whether a given piece of wire will be stable if placed in a vertical position with its lower end clamped.

(See Greenhill, *Proc. Camb. Phil. Soc.* IV. 1881, and Love, *Math. Theory of Elasticity*, II. p. 297.)

As another simple illustration derived from the theory of elasticity, we will give, after Pochhammer and Love (*l.c.* p. 115), a short discussion of the torsional vibration of an isotropic solid circular cylinder of radius c.

If (r, θ, z) are the coordinates of any point of the cylinder, and u, v, w the corresponding displacements, the equations of motion for small vibrations are

$$\left.\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + 2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial \varpi_3}{\partial \theta} + 2\mu \frac{\partial \varpi_2}{dz} \\ \rho \frac{\partial^2 v}{\partial t^2} &= (\lambda + 2\mu)\frac{1}{r}\frac{\partial \Delta}{\partial \theta} - 2\mu \frac{\partial \varpi_1}{\partial z} + 2\mu \frac{\partial \varpi_3}{\partial r} \\ \rho \frac{\partial^2 w}{\partial t^2} &= (\lambda + 2\mu)\frac{\partial \Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_2) + \frac{2\mu}{r}\frac{\partial \varpi_1}{\partial \theta} \end{aligned}\right\},$$

where

$$\left.\begin{aligned} \Delta &= \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}, \\ 2\varpi_1 &= \frac{1}{r}\left(\frac{\partial w}{\partial \theta} - \frac{\partial (rv)}{\partial z}\right), \\ 2\varpi_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \\ 2\varpi_3 &= \frac{1}{r}\left(\frac{\partial (rv)}{\partial r} - \frac{\partial u}{\partial \theta}\right) \end{aligned}\right\}.$$

The stresses across a cylindrical surface $r = \text{constant}$ are

$$\left.\begin{aligned} P_{rr} &= \lambda\Delta + 2\mu \frac{\partial u}{\partial r}, \\ P_{\theta r} &= \mu \left\{\frac{1}{r}\frac{\partial u}{\partial \theta} - r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}, \\ P_{zr} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \end{aligned}\right\}.$$

We proceed to construct a particular solution of the type

$$u = 0, \quad w = 0, \quad v = Ve^{i(\gamma z + pt)},$$

where V is a function of r only, and γ, p are constants. In order to obtain a periodic vibration with no damping, we suppose that p is real. The torsional character of the oscillation is clear from the form of u, v, w.

If we put, for the moment,

$$e^{i(\gamma z+pt)}=Z,$$

we have

$$\Delta=0,$$
$$2\varpi_1=-i\gamma VZ,$$
$$2\varpi_2=0,$$
$$2\varpi_3=\frac{Z}{r}\frac{\partial(rV)}{\partial r},$$

and the equations of motion reduce to two identities and

$$-\rho p^2VZ=-\mu\gamma^2VZ+\mu Z\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rV)\right).$$

Thus V must satisfy the equation

$$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}+\left\{\frac{\rho p^2-\mu\gamma^2}{\mu}-\frac{1}{r^2}\right\}V=0,$$

and since V must be finite when $r=0$, the proper solution is

$$V=AJ_1(\kappa r)$$

where

$$\kappa^2=\frac{\rho}{\mu}p^2-\gamma^2.$$

If the curved surface of the cylinder is free, then the stresses P_{rr}, $P_{\theta r}$, P_{zr} must vanish when $r=c$. Now P_{rr} and P_{zr} vanish identically; $P_{\theta r}$ will vanish if

$$\frac{d}{dr}\left\{\frac{J_1(\kappa r)}{r}\right\}=0,$$

when $r=c$: that is, if

$$\kappa cJ_1'(\kappa c)-J_1(\kappa c)=0,$$

or, which is the same thing, if

$$\kappa cJ_2(\kappa c)=0,$$

(see pp. 13, 179).

If $\kappa=0$, the differential equation to find V is

$$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{1}{r^2}V=0,$$

of which the solution is

$$V=Ar+\frac{B}{r},$$

or, for our present purpose

$$V = Ar.$$

This leads to a solution of the original problem in the shape

$$u = 0, \quad w = 0,$$

$$v = Are^{i(\gamma z + pt)},$$

with

$$\rho p^2 - \mu\gamma^2 = 0;$$

and in particular we have as a special case

$$v = r\sum_n {\cos \atop \sin} \frac{n\pi z}{l}\left(A_n \cos\frac{n\pi t}{l}\sqrt{\frac{\mu}{\rho}} + B_n \sin\frac{n\pi t}{l}\sqrt{\frac{\mu}{\rho}}\right),$$

when $n = 0, 1, 2, \ldots$, and l is a constant.

But we may also take for κ any one of the real roots of

$$J_2(\kappa c) = 0$$

of which there is an infinite number. If κ_s is any one of these, p any real constant, and γ determined, as a real or pure imaginary constant, by the equation

$$\gamma^2 = \frac{\rho}{\mu}p^2 - \kappa_s^2,$$

we have a solution

$$v = AJ_1(\kappa_s r)\, e^{i(\gamma z + pt)}.$$

Unless some further conditions are assigned, p is arbitrary, whatever value of κ_s is taken: that is to say, vibrations of any period are possible.

When the period is $2\pi/p$ the velocity of propagation parallel to the axis of the cylinder is

$$\frac{p}{\gamma} = \frac{p\sqrt{\mu}}{\sqrt{\rho p^2 - \mu\kappa_s^2}},$$

which is approximately equal to $\sqrt{\mu/\rho}$ so long as $\mu\kappa_s^2$ is small compared with ρp^2.

If $\rho p^2 - \mu\kappa_s^2$ is negative the type of vibration is altered: there is now a damping of the vibration as we go in one direction along the axis of the cylinder.

Special solutions may be constructed to suit special boundary conditions: thus for instance if we put

$$v = \sum_{s,m} (A_{sm} \cos pt + B_{sm} \sin pt)\, J_1(\kappa_s r) \sin \frac{m\pi z}{l},$$

when m is a real integer, κ_s any root of $J_2(\kappa_s c) = 0$, and

$$p^2 = \frac{\mu}{\rho}\left\{\kappa_s^2 + \frac{m^2\pi^2}{l^2}\right\},$$

this gives a possible mode of vibration for a cylinder of radius c and length $2l$, the circular ends of which are glued to fixed parallel planes, the curved surface of the cylinder being left free. The doubly infinite number of constants A_{sm}, B_{sm} have to be determined by suitable initial conditions.

For the discussion of the extensional and flexural vibrations the reader should consult Love's treatise already referred to, and the memoir of Pochhammer, *Crelle*, lxxxi. p. 324.

Many other illustrations of the use of Bessel functions in the theory of elasticity will be found in recent memoirs by Chree, Lamb, Love, Rayleigh and others.

To conclude, as we have begun, with the oscillations of a chain, let us modify Bernoulli's problem by supposing that the density at any point of the chain varies as the nth power of its distance from the lower end.

Proceeding as on p. 1, but measuring x from the free end, the equation of motion is

$$x^n \frac{d^2y}{dt^2} = \frac{d}{dx}\left(\frac{gx^{n+1}}{n+1}\frac{dy}{dx}\right),$$

and if we put

$$y = u \cos 2\pi pt,$$

where u is a function of x, we have

$$\frac{gx}{n+1}\frac{d^2u}{dx^2} + g\frac{du}{dx} + 4\pi^2p^2u = 0,$$

or

$$\frac{d^2u}{dx^2} + \frac{n+1}{x}\frac{du}{dx} + \frac{\kappa^2}{x}u = 0,$$

where

$$\kappa = 2\pi p\sqrt{(n+1)/g}.$$

Assuming

$$u = a_0 + a_1x + a_2x^2 + \dots,$$

we find that the differential equation is satisfied by

$$u = a_0\left\{1 - \frac{\kappa^2 x}{n+1} + \frac{\kappa^4 x^2}{2(n+1)(n+2)} - \frac{\kappa^6 x^3}{2.3(n+1)(n+2)(n+3)} + \dots\right\},$$

or, which is the same thing, by

$$u = Ax^{-\frac{1}{2}n} J_n(2\kappa x^{\frac{1}{2}}).$$

Finally, therefore,

$$y = Ax^{-\frac{1}{2}n} J_n\left\{4\pi p\sqrt{\frac{(n+1)x}{g}}\right\}\cos 2\pi pt.$$

Professor Greenhill, to whom this extension of Bernoulli's problem is due, remarks that to realise the conditions of the problem practically, we should take, instead of the chain, a blind composed of a very large number of small uniform horizontal rods, the shape of the blind being defined by the curves

$$c^{n-1}y = \pm x^n,$$

with x positive.

Thus $n = 1$ gives a triangular blind, and so on.

In connection with the reduction of the differential equation at p. 216, it may be pointed out here, that, if $yr^{-\lambda}$ be substituted for u, and $x^{\frac{1}{\mu}}\kappa^{-1}$ for r, the differential equation of the form

$$\frac{d^2u}{dr^2} + (2\lambda + 1)\frac{1}{r}\frac{du}{dr} + (\kappa^2\mu^2 r^{2\mu} + \lambda^2 - \mu^2 n^2)\frac{1}{r^2}u = 0,$$

can be reduced to the standard form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0,$$

so that it is integrable by Bessel functions.

This gives a general rule for the transformation in cases in which it is not so obvious as in the case considered above. In that case

$$u = p,\ \lambda = -\tfrac{1}{2},\ n = \pm\tfrac{1}{3},\ \mu = 1,\ \kappa^2 = 4w/9\beta.$$

(See p. 233 below for other examples.)

NOTE.

As the determination of the coefficients of $Y_n(x)$, $J_n(x)$ in that second solution of the general differential equation which vanishes at infinity is not without difficulty, and the solution is of great importance for physical applications, the following explanation of Weber's treatment of the problem (*Crelle*, Bd. 75, 1873) may not be superfluous.

It has been proved, pp. 59, 60 above, that if z be complex, and such that the real part of zi is negative

$$z^n \int_{-1}^{+1} e^{z\lambda i} (1-\lambda^2)^{n-\frac{1}{2}}\, d\lambda = 2^n \sqrt{\pi}\, \Pi\left(n-\tfrac{1}{2}\right) J_n(z). \qquad \text{I}$$

Further the investigation given in Chapter VII. above shows that the differential equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{n^2}{z^2}\right) w = 0$$

is satisfied also by taking

$$w = z^n \int_1^\infty e^{iz\lambda} (\lambda^2 - 1)^{n-\frac{1}{2}}\, d\lambda. \qquad 2$$

For by differentiation it will be found that the quantity on the left in the differential equation reduces to

$$-iz^{n-1}\left\{e^{iz\lambda} (\lambda^2-1)^{n+\frac{1}{2}}\right\}_1^\infty$$

which vanishes when $\lambda = 1$, and when $\lambda = \infty$.

These two definite integral solutions of the differential equation seem to be due to Riemann, who gave them in his memoir on Nobili's Rings referred to on p. 128 above.

In the second of these solutions, on the supposition that the path of integration is along the axis of real quantity from 1 to ∞, the real part of iz must be negative, so that, either z must be essentially complex, or $n < 0$. The solution then in the absence of fulfilment of the latter condition does not hold for real values of z. We can find a solution which holds for both real and complex values of z as follows.

Let a new path of integration from $+1$ to 0, and from 0 to ∞i be chosen. We have

$$w = z^n \left\{ \int_{+1}^{0} e^{iz\lambda} (\lambda^2 - 1)^{n-\frac{1}{2}} d\lambda + \int_0^{\infty i} e^{iz\lambda} (\lambda^2 - 1)^{n-\frac{1}{2}} d\lambda \right\}. \qquad 3$$

Now, for the moment let z be real, $= x$, say, and we have putting in the second integral μi for λ, where μ is real

$$w = (-x)^n \left\{ \frac{1}{i} \int_{+1}^{0} e^{ix\lambda} (1 - \lambda^2)^{n-\frac{1}{2}} d\lambda + \int_0^{\infty} e^{-x\mu} (1 + \mu^2)^{n-\frac{1}{2}} d\mu \right\}: \qquad 4$$

Since λ is real in the first integral we may use λ as variable in the second part also, and we have

$$\text{Real part of } w = (-x)^n \left\{ \int_0^{\infty} e^{-x\lambda} (1 + \lambda^2)^{n-\frac{1}{2}} d\lambda - \int_0^1 \sin(x\lambda) (1 - \lambda^2)^{n-\frac{1}{2}} d\lambda \right\}.$$

Thus if

$$u = x^n \left\{ \int_0^1 \sin(x\lambda)(1 - \lambda^2)^{n-\frac{1}{2}} d\lambda - \int_0^{\infty} e^{-x\lambda} (1 + \lambda^2)^{n-\frac{1}{2}} d\lambda \right\}, \qquad 5$$

u is a solution of the general differential equation. It will hold also for complex values of x provided that the real part of x is positive.

We now write

$$\frac{u}{1.3.5\ldots(2n-1)} = A_n Y_n(x) + B_n J_n(x),$$

where A_n, B_n are constants to be determined. This can be done as follows. Multiplying both sides by x^n, and putting $x = 0$, we have zero for the second term on the right, and therefore

$$\operatorname*{Lt}_{x=0} \frac{x^n u}{1.3\ldots(2n-1)} = \operatorname*{Lt}_{x=0} A_n Y_n(x).$$

The first term on the right of 5 multiplied by x^n vanishes when $x = 0$. Expanding the quantity under the sign of integration in the second term by Taylor's theorem we obtain the expression

$$x^n \int_0^{\infty} e^{-x\lambda} \left\{ \lambda^{2n-1} + \frac{2n-1}{2} \lambda^{2n-3} + \ldots + \lambda^{2n-1} R \right\} d\lambda,$$

where

$$R = \frac{(2n-1)(2n-3)\ldots3}{2^{n-1}(n-1)!} \left(1 + \frac{\theta}{\lambda^2}\right)^{\frac{1}{2}} \frac{1}{\lambda^{2(n-1)}}$$

$(0 < \theta < 1)$. Integrated this takes the form

$$x^n \left\{ \frac{\Gamma(2n)}{x^{2n}} + \frac{2n-1}{2} \frac{\Gamma(2n-1)}{x^{2n-1}} + \ldots \right\},$$

and the limit when $x = 0$ of $x^n u$ is therefore $-\Gamma(2n)$, that is $-(2n-1)!$.

Now from the explicit form of $Y_n(x)$, 30, p. 14 above, we see that the limit of $x^n Y_n(x)$ when $x=0$ is $-2^{n-1}(n-1)!$. Hence finally

$$\operatorname*{Lt}_{x=0} \frac{x^n u}{1 \cdot 3 \ldots (2n-1)} = -\frac{(2n-1)!}{1 \cdot 3 \ldots (2n-1)} = -A_n 2^{n-1}(n-1)!$$

or
$$A_n = 1. \qquad 6$$

Writing u_n for what we have called u, we have

$$\frac{d}{dx}\frac{u_n}{x^n} = \int_0^1 \xi \cos(x\xi) \cdot (1-\xi)^{2\frac{2n-1}{2}} d\xi + \int_0^\infty \xi e^{-x\xi}(1+\xi^2)^{\frac{2n-1}{2}} d\xi.$$

Integrating by parts we get

$$\frac{d}{dx}\frac{u_n}{x^n} = \frac{1}{2n+1} - \frac{1}{2n+1} - \frac{x}{2n+1}\int_0^1 \sin(x\xi) \cdot (1-\xi^2)^{\frac{2n+1}{2}} d\xi$$
$$+ \frac{x}{2n+1}\int_0^\infty e^{-x\xi}(1+\xi^2)^{\frac{2n+1}{2}} d\xi$$
$$= -x\frac{u_{n+1}}{x^{n+1}};$$

or
$$\frac{u_{n+1}}{x^{n+1}} + \frac{1}{x}\frac{d}{dx}\frac{u_n}{x^n} = 0.$$

Thus u_{n+1} is derived from u_n without any change of coefficient, and so comparing with the case of $n=0$ (see Examples 15...19, p. 229 below) we get

$$B = \gamma - \log 2. \qquad 7$$

Thus for the values of x specified the solution of the general differential equation may be written by 6 and 7,

$$u_n = C\{Y_n(x) + (\gamma - \log 2) J_n(x)\}, \qquad 8$$

where C is a constant.

EXAMPLES.

1. If, as on p. 4, we put

$$\mu = \phi - e \sin \phi,$$

prove that

$$\frac{\partial^2 \phi}{\partial e^2} + \frac{1}{e}\frac{\partial \phi}{\partial e} + \frac{1-e^2}{e^2}\frac{\partial^2 \phi}{\partial \mu^2} = 0,$$

and hence obtain Bessel's expression for ϕ in terms of μ.

2. Prove that, in the problem of elliptic motion, the radius vector SP is given by the equation

$$\frac{a}{r} = \frac{1}{1 - e\cos\phi}$$

$$= 1 + 2\{J_1(e)\cos\mu + J_2(2e)\cos 2\mu + \ldots\},$$

(see p. 19).

3. Verify the following expansions :—

(i) $e^{nx} = J_0(x) + \sum_1^\infty \{(n + \sqrt{n^2+1})^s + (n - \sqrt{n^2+1})^s\} J_s(x).$

(ii) $\cosh nx = J_0(x) + 2\Sigma \cosh s\phi J_s(x),$ $\qquad [s = 2, 4, \ldots]$

$\sinh nx = 2\Sigma \sinh s\phi J_s(x),$ $\qquad [s = 1, 3, \ldots]$

where $\qquad \phi = \sinh^{-1} n.$

(iii) $\cos nx = J_0(x) + 2\Sigma (-)^{\frac{1}{2}s} \cosh s\phi J_s(x),$ $\qquad [s = 2, 4, \ldots]$

$\sin nx = 2\Sigma (-)^{\frac{1}{2}(s-1)} \cosh s\phi J_s(x),$ $\qquad [s = 1, 3, \ldots]$

where $\phi = \cosh^{-1} n$, n being supposed greater than 1.

(iv) $Y_2 = J_2 \log x - \frac{8}{x^3} - \frac{1}{x} - \frac{1}{4}J_1 - \frac{5}{3}J_3 + \ldots$

$$+ \frac{3(4s+1)}{2(2s+1)(2s+2)} J_{4s+1} - \frac{4s+3}{4s(2s+3)} J_{4s+3} + \ldots. \qquad [s = 1, 2, \ldots]$$

4. Prove that

$$1 = J_0^2 + 2J_1^2 + 2J_2^2 + \ldots,$$

$$J_1^2 = 2\,(J_0J_2 + J_1J_3 + J_2J_4 + \ldots),$$

$$2J_1J_3 - J_2^2 = 2\,(J_0J_4 + J_1J_5 + J_2J_6 + \ldots).$$

5. Show that

$$bcJ_1(\sqrt{b^2+c^2}) = 2\sqrt{b^2+c^2}\,\{J_1(b)\,J_1(c) - 3J_3(b)\,J_3(c) + 5J_5(b)\,J_5(c) - \ldots\}.$$

6. Prove that

$$e^{-t}I_n(t) = \frac{t^n}{2^n \cdot n!}\left\{1 - t + \frac{2n+3}{2\,(2n+2)}\,t^2 - \frac{2n+5}{2\,.\,3\,(2n+2)}\,t^3 + \frac{(2n+5)\,(2n+7)}{2\,.\,3\,.\,4\,.\,(2n+2)\,(2n+4)}\,t^4 - \frac{(2n+7)\,(2n+9)}{2\,.\,3\,.\,4\,.\,5\,.\,(2n+2)\,(2n+4)}\,t^5 + \ldots\right\}.$$

7. Verify the following results (taken from Basset's Hydrodynamics, vol. II.)

$$K_0(x) = \int_0^\infty \frac{aJ_0(ax)\,da}{1+a^2},$$

$$\int_0^\infty K_0(ax)\cos bx\,dx = \tfrac{1}{2}\pi\,(a^2+b^2)^{-\frac{1}{2}},$$

$$\int_0^\infty e^{-ax}K_0(bx)\,dx = (b^2-a^2)^{-\frac{1}{2}}\tan^{-1}\frac{\sqrt{b^2-a^2}}{a};\ b > a,$$

$$= (a^2-b^2)^{-\frac{1}{2}}\tanh^{-1}\frac{\sqrt{a^2-b^2}}{a};\ b < a,$$

$$\int_0^\infty K_0(ax)\,J_0(bx)\,dx = (a^2+b^2)^{-\frac{1}{2}}\,F\{b\,(a^2+b^2)^{-\frac{1}{2}}\},$$

$$\int_0^\infty e^{-ax}\{J_0(bx)\}^2\,dx = 2\pi^{-1}(a^2+4b^2)^{-\frac{1}{2}}\,F\{2b\,(a^2+4b^2)^{-\frac{1}{2}}\}.$$

[$F(k)$ denotes the first complete elliptic integral to modulus k.]

8. Prove that

$$Y_0(x) = \frac{2}{\pi}\int_0^\infty \cos(x\cosh\theta)\,d\theta,$$

$$\int_0^\infty e^{-a^2x^2}\,Y_0(bx)\,dx = \frac{1}{2a\sqrt{\pi}}\,e^{-\frac{b^2}{8a^2}}K_0\left(\frac{b^2}{8a^2}\right),$$

$$\int_0^\infty e^{-a^2x^2}\,K_0(bx)\,dx = \frac{\sqrt{\pi}}{4a}\,e^{\frac{b^2}{8a^2}}K_0\left(\frac{b^2}{8a^2}\right).$$

(Basset.)

[The following Examples (9—19) are taken from Weber's paper in *Crelle*, Bd. 75, 1873.]

9. **Prove that**

$$J_1(x) = -\frac{i}{\pi}\int_0^\pi e^{ix\cos\theta}\cos\theta\, d\theta.$$

10. **Prove that**

$$\begin{aligned}\int_0^\infty J_1(x)\, J_0(ax)\, dx &= 1, \quad (a^2 > 1)\\ &= \tfrac{1}{2}, \quad (a^2 = 1)\\ &= 0, \quad (a^2 > 1).\end{aligned}$$

[Substitute the value of $J_1(x)$ from Ex. 9, and integrate first with respect to x.]

11. **Prove that**

$$\int_1^\infty \frac{e^{-\epsilon\lambda}\sin(\lambda x)}{\sqrt{\lambda^2-1}}\, d\lambda = \int_0^\infty e^{-\epsilon\lambda}\sin(\lambda x)\, d\lambda \int_0^\infty \sin(\lambda r)\, J_0(r)\, dr.$$

Hence show by integrating first with respect to λ on the right and having regard to 151, p. 73, that

$$J_0(x) = \frac{2}{\pi}\int_1^\infty \frac{\sin(\lambda x)}{\sqrt{\lambda^2-1}}\, d\lambda,$$

and therefore

$$\int_0^1 \frac{\sin(\lambda x)}{\sqrt{1-\lambda^2}}\, d\lambda = \int_1^\infty \frac{\sin(\lambda x)}{\sqrt{\lambda^2-1}}\, d\lambda.$$

12. **Prove that**

$$\int_0^\infty \log x\, J_0(x)\, dx = -(\gamma + \log 2),$$

where $$\gamma = -\int_0^\infty e^{-x}\log x\, dx, \text{ (see p. 40 above).}$$

[By the last example,

$$\int_0^\infty e^{-\epsilon x}\log x J_0(x)\, dx = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda^2-1}}\int_0^\infty e^{-\epsilon x}\log x\sin(\lambda x)\, dx,$$

and the second integral on the right can be evaluated by known theorems. Or, more simply, the value of $\log x$ given by the equation

$$\log x = \int_0^\infty \frac{e^{-u} - e^{-ux}}{u}\, du$$

may be substituted in the given integral, and the integration performed first with respect to x then with respect to u.]

13. Prove that

$$\int_0^\infty Y_0(x)\,dx = \log 2 - \gamma.$$

[Use Neumann's series for Y_0, p. 22 above, and the theorem, 149, p. 73, with the result of the preceding example.]

14. Establish the equations

$$\left.\begin{aligned}
(a)\quad & \int_0^\infty \frac{\sin(a\lambda)\,J_0(\lambda x)\,d\lambda}{k^2+\lambda^2} = \frac{\sinh ka}{k}\int_1^\infty \frac{e^{-kx\lambda}}{\sqrt{\lambda^2-1}}\,d\lambda \\
(b)\quad & \int_0^\infty \frac{\lambda\cos(a\lambda)\,J_0(\lambda x)}{k^2+\lambda^2}\,d\lambda = \cosh ka\int_1^\infty \frac{e^{-kx\lambda}}{\sqrt{\lambda^2-1}}\,d\lambda
\end{aligned}\right\}\ (x>a)$$

$$\left.\begin{aligned}
(c)\quad & \int_0^\infty \frac{\cos(a\lambda)\,J_0(\lambda x)}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2}\frac{e^{-ka}}{k}\,J_0(ikx) \\
(d)\quad & \int_0^\infty \frac{\lambda\sin(a\lambda)\,J_0(\lambda x)}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-ka}\,J_0(ikx)
\end{aligned}\right\}\ (x<a).$$

[Use the theorems

$$\int_0^\infty \frac{\cos x\lambda}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2k}e^{-kx},\quad \int_0^\infty \frac{\lambda\sin(x\lambda)}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-kx}.]$$

15. Prove that the definite integrals

$$\int_{-1}^{+1} \frac{e^{iz\lambda}}{\sqrt{1-\lambda^2}}\,d\lambda,\quad \int_1^\infty \frac{e^{iz\lambda}}{\sqrt{\lambda^2-1}}\,d\lambda$$

(in which z may be real or imaginary, but is always such that the real part, if any, of iz is negative) both satisfy the differential equation of the Bessel function of zero order.

16. If (see Ex. 15)

$$f(z) = \frac{2}{\pi}\int_1^\infty \frac{e^{iz\lambda}}{\sqrt{\lambda^2-1}}\,d\lambda,$$

prove that for z real and positive ($=x$, say)

$$f(x) = \frac{2}{\pi}\int_1^\infty \frac{\cos(x\lambda)}{\sqrt{\lambda^2-1}}\,d\lambda + iJ_0(x),$$

and for z real and negative ($=-x$),

$$f(-x) = \frac{2}{\pi}\int_1^\infty \frac{\cos(x\lambda)}{\sqrt{\lambda^2-1}} - iJ_0(x).$$

17. If we write, as from Ex. 15 we are entitled to do,

$$f(z) = A_0 Y_0(z) + BJ_0(z),$$

prove that
$$f(-x) - f(x) = A\pi i J_0(x),$$

and therefore
$$A = -\frac{2}{\pi}$$

by the theorem of Ex. 16.

[Change from $+x$ to $-x$ along a semicircle round the origin and have regard to the term $J_0(x)\log x$ in $Y_0(x)$.]

18. Prove that

$$\int_0^\infty f(x)\,dx = i,$$

and hence by Ex. 13 and the theorem 149, p. 73, that

$$B = i - \frac{2}{\pi}(\gamma - \log 2),$$

so that

$$\int_1^\infty \frac{e^{iz\lambda}}{\sqrt{\lambda^2 - 1}}\,d\lambda = -\{Y_0(z) + (\gamma - \log 2) J_0(z)\} + \frac{i\pi}{2} J_0(z).$$

19. Prove that

$$\int_0^\infty \frac{\lambda J_0(\lambda z)}{k^2 + \lambda^2}\,d\lambda = -\{Y_0(ikz) + (\gamma - \log 2) J_0(ikz) + \frac{i\pi}{2} J_0(ikz).$$

20. Prove that if u is a function of x and y which satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0,$$

and which, as well as its derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, is finite and continuous for all points within and upon the circle

$$x^2 + y^2 - r^2 = 0,$$

then
$$\int_0^{2\pi} u\,d\phi = 2\pi u_0 J_0(\kappa r),$$

when the integral is taken along the circumference of the circle, and u_0 is the value of u at the origin.

(Weber, *Math. Ann.* i. p. 9.)

21. Prove that if

$$V = 2\pi^{-1}\int_0^\infty d\mu \int_0^c e^{-\mu z}\cos\lambda v \cos\mu v J_0(\mu\varpi)\, av,$$

then

$$V = J_0(\lambda\varpi) \text{ when } z = 0 \text{ and } \varpi < c,$$

and

$$\frac{\partial V}{\partial z} = 0 \qquad \text{when } z = 0 \text{ and } \varpi > c.$$

Show also that if

$$V = 2\pi^{-1}\int_0^\infty d\mu \int_0^c e^{-\mu z}\sin\lambda v \sin\mu v J_1(\mu\varpi)\, dv,$$

then

$$V = J_1(\lambda\varpi) \text{ when } z = 0 \text{ and } \varpi < c,$$

$$\frac{\partial V}{\partial z} = 0 \qquad \text{when } z = 0 \text{ and } \varpi > c.$$

(Basset, *Hydrodynamics*, II. p. 33.)

22. If $\eta = (1 + e^2)^{\frac{1}{2}} \sin\xi$ be the equation to a curve referred to oblique axes inclined to one another at an angle $\cot^{-1} e$, show that the equation of the curve referred to rectangular axes, with the axis of x coinciding with that of ξ, is

$$y = \sum_1^\infty (-)^{n+1} \frac{2}{ne} J_n(ne) \sin nx.$$

23. If

$$x^4 - b^4 = \Sigma L_n J_0(nx),$$

the summation extending to all values of n given by $J_0(nb) = 0$, then

$$L_n = \frac{2}{b^2\{J_1(nb)\}^2}\int_0^b (x^4 - b^4)\, xJ_0(nx)\, dx$$

$$= \frac{8b}{n^3}\,\frac{2J_3(nb) - nbJ_2(nb)}{\{J_1(nb)\}^2} = \frac{32}{b}\,\frac{4 - n^2b^2}{n^5 J_1(nb)}.$$

24. Prove that if $n = k + \frac{1}{2}$, where k is zero or a real integer,

$$J_n^2(x) + J_{-n}^2(x)$$

is a rational integral function of x^{-1}.

For instance

$$J_{-\frac{1}{2}}^2 + J_{\frac{1}{2}}^2 = \frac{2}{\pi x},$$

$$J_{-\frac{3}{2}}^2 + J_{\frac{3}{2}}^2 = \frac{2}{\pi x}\left(1 + \frac{1}{x^2}\right),$$

and so on. (Lommel.)

25. Prove that

$$\frac{\sin 2x}{\pi} = J_{\frac{1}{2}}^2 - 3J_{\frac{3}{2}}^2 + 5J_{\frac{5}{2}}^2 - \dots.$$

(Lommel.)

26. Prove that if D denote $\dfrac{d}{dx}$, then

$$D^m \{x^{-\frac{1}{2}n} J_n(\sqrt{x})\} = (-\tfrac{1}{2})^m x^{-\frac{1}{2}(n+m)} J_{n+m}(\sqrt{x}),$$

$$D^m \{x^{\frac{1}{2}n} J_n(\sqrt{x})\} = (\tfrac{1}{2})^m x^{\frac{1}{2}(n-m)} J_{n-m}(\sqrt{x}).$$

(Lommel.)

27. Prove that the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + V = 0$$

is satisfied by

$$V = (A_m \cos m\phi + B_m \sin m\phi)(-2\rho)^m \frac{d^m J_0(\rho)}{d(\rho^2)^m},$$

where $x = \rho\cos\phi$, $y = \rho\sin\phi$.

Show also that

$$\left\{P_0(\mu) + \frac{1}{2!}P_2(\mu) + \frac{1}{4!}P_4(\mu) + \dots\right\}^2$$

$$- \left\{P_1(\mu) + \frac{1}{3!}P_3(\mu) + \frac{1}{5!}P_5(\mu) + \dots\right\}^2 = \{J_0(\sqrt{1-\mu^2})\}^2,$$

and obtain a corresponding expression for $\{J_m(\sqrt{1-\mu^2})\}^2$.

28. Show that the equation

$$\frac{d^2R}{dx^2} + \frac{2}{x}\frac{dR}{dx} + R = \frac{n(n+1)}{x^2}R,$$

is satisfied by either of the series

$$u_n = \frac{(-1)^n x^n}{1.3\dots(2n+1)}\left\{1 - \frac{1}{1.(2n+3)}\frac{x^2}{2} + \frac{1}{1.2.(2n+3)(2n+5)}\frac{x^4}{4} - \dots\right\},$$

$$v_n = \frac{(-1)^n 1.3\dots(2n-1)}{x^{n+1}}\left\{1 + \frac{1}{1.(2n-1)}\frac{x^2}{2} + \frac{1}{1.2.(2n-1)(2n-3)}\frac{x^4}{4} + \dots\right\}.$$

Express u_n as a Bessel function; and show that

$$u_n u_{-(n+1)} = v_n v_{-(n+1)}.$$

29. Verify the following solutions of differential equations by means of Bessel functions :—

(i) If

$$x^{m+\frac{1}{2}}\frac{d^{2m+1}y}{dx^{2m+1}} \mp y = 0,$$

then

$$y = x^{\frac{2m+1}{4}} \sum_0^{2m} C_p \{J_{-m-\frac{1}{2}}(2a_p\sqrt{x}) + iJ_{m+\frac{1}{2}}(2a_p\sqrt{x})\},$$

when $a_0, a_1 \dots a_{2m}$ are the roots of $a^{2m+1} = \mp i$, and $C_0, C_1, \dots C_{2m}$ are arbitrary constants.

(ii) If

$$\frac{d^2y}{dx^2} - \frac{2n-1}{x}\frac{dy}{dx} + y = 0,$$

then

$$y = x^n [AJ_n(x) + BJ_{-n}(x)].$$

(iii) If

$$x^2\frac{d^2y}{dx^2} - (2n\beta - 1)\,x\frac{dy}{dx} + \beta^2\gamma^2x^{2\beta}y = 0,$$

then

$$y = x^{\beta n}[AJ_n(\gamma x^\beta) + BJ_{-n}(\gamma x^\beta)].$$

(iv) If

$$x^2\frac{d^2y}{dx^2} + (2a - 2\beta n + 1)\,x\frac{dy}{dx} + \{a(a - 2\beta n) + \beta^2\gamma^2x^{2\beta}\}\,y = 0,$$

then

$$y = x^{\beta n - a}[AJ_n(\gamma x^\beta) + BJ_{-n}(\gamma x^\beta)].$$

(v) Deduce from (iv) that if

$$\frac{d^2y}{dx^2} + x^{2\beta-2}y = 0,$$

(a form of Riccati's equation)

$$y = \sqrt{x}\left[AJ_{\frac{1}{2\beta}}\left(\frac{x^\beta}{\beta}\right) + BJ_{-\frac{1}{2\beta}}\left(\frac{x^\beta}{\beta}\right)\right],$$

and solve

$$\frac{d^2y}{dx^2} - x^{2\beta-2}y = 0.$$

(Lommel.)

30. Prove that if u is any integral of

$$\frac{d^2u}{dx^2} + Xu = 0,$$

when X is a function of x, and if

$$\psi = a\int \frac{dx}{u^2} + b,$$

where a, b are constants, then the complete integral of

$$\frac{d^2y}{dx^2} + \left\{X + \frac{a^2}{u^4}\left[1 - (n^2 - \tfrac{1}{4})\,\psi^{-2}\right]\right\} y = 0$$

is

$$y = u\sqrt{\psi}\,\{AJ_n(\psi) + BJ_{-n}(\psi)\}.$$

(Lommel.)

31. Prove that the solution of Riccati's equation

$$x\frac{dy}{dx} - ay + by^2 = cx^p$$

can be made to depend upon the solution of Bessel's equation

$$r^2\frac{d^2w}{dr^2} + r\frac{dw}{dr} + (k^2r^2 - n^2)\,w = 0,$$

where $n = a/p$.

32. If a bead of mass M be attached to the lowest end of a uniform flexible chain hanging vertically, then the displacement at a point of the chain distant s from the fixed end is, for the small oscillations about the vertical,

$$\sum_n (A_n \cos nt + B_n \sin nt)\, V_n,$$

where

$$V_n = \{n\sqrt{M}\,Y_0(n\beta\sqrt{M}) - \sqrt{mg}\,Y_1(n\beta\sqrt{M})\}\,J_0(n\beta\sqrt{\mu - ms})$$
$$- \{n\sqrt{M}\,J_0(n\beta\sqrt{M}) - \sqrt{mg}\,J_1(n\beta\sqrt{M})\}\,Y_0(n\beta\sqrt{\mu - ms}),$$

μ being the total mass of the chain and bead, and β denoting $2/\sqrt{mg}$, where m is the mass of unit length of the chain. How are the values of n to be determined?

33. Assuming that $J_0(x)$ vanishes when $x = 2{\cdot}4$, show that in a V-shaped estuary 53 fathoms ($10{,}000 \div 32{\cdot}2$ ft.) deep, which communicates with the ocean, there will be no semi-diurnal tide at about 300 miles from the end of the estuary.

(See p. 113 *et seq.*)

34. The initial temperature of a homogeneous solid sphere of radius a is given by

$$v_0 = Ar^{-2}\cos\theta\,(\sin mr - mr\cos mr):$$

prove that at time t its temperature is

$$u = v_0 e^{-m^2kt},$$

provided that m is a root of the equation

$$(ah - 2k)(ma\cot ma - 1) = m^2a^2k,$$

k, h being the internal and surface conductivities, and the surrounding medium being at zero temperature.

(Weber.)

35. A spherical bell of radius c is vibrating in such a manner that the normal component of the velocity at any point of its surface is $S_n \cos kat$, where S_n is a spherical surface harmonic of degree n, and a is the velocity of transmission of vibrations through the surrounding air. Prove that the velocity potential at any point outside the bell at a distance r from the centre, due to the disturbance propagated in the air outwards, is the real part of the expression

$$-\frac{c^2}{r} e^{ik(at-r+c)} \frac{f_n(ikr)\,S_n}{(1+ikc)f_n(ikc) - ikcf'(ikc)},$$

where

$$f_n(x) = (-)^n xe^x P_n\left(\frac{d}{dx}\right)\frac{e^{-x}}{x},$$

P_n denoting the zonal harmonic of degree n.

Show that the resultant pressure of the air on the bell is zero except when $n = 1$.

A sphere is vibrating in a given manner as a rigid body about a position of equilibrium which is at a given distance from a large perfectly rigid obstacle whose surface is plane; determine the motion at any point in the air.

36. A sector of an infinitely long circular cylinder is bounded by two rigid planes inclined at an angle 2α, and is closed at one end by a flexible membrane which is forced to perform small normal oscillations, so that the velocity at any point, whose coordinates, referred to the centre as origin and the bisector of the angle of the sector as initial line, are r, θ, is $qr^p \cos p\theta \cos nct$, where $p\alpha = i\pi$, i being an integer and c the velocity of propagation of plane waves in air. Prove that, at time t, the velocity potential at any point (r, θ, z) of the air in the cylinder is

$$2qpa^p \cos p\theta \,\Sigma \frac{1}{k}\frac{J_p(n'r)}{(n'^2a^2 - p^2)\,J_p(n'a)} \cos nct\,\{e^{-kz} \ \textit{or}\ \sin kz\},$$

where $J_p'(n'a) = 0$ gives the requisite values of n', a being the radius of the cylinder, and where k is a real quantity given by the equation $n'^2 = n^2 \pm k^2$, the upper and lower sign before k^2 corresponding to the first and second term in the bracket respectively.

37. A given mass of air is at rest in a circular cylinder of radius c under the action of a constant force to the axis. Show that if the force suddenly cease to act, then the velocity function at any subsequent time varies as

$$\Sigma \frac{1}{k^2} \frac{J_0(kr)}{J_0(kc)} \sin kat,$$

where a is the velocity of sound in air, the summation extends to all values of k satisfying $J_1(kc) = 0$, and the square of the condensation is neglected.

38. A right circular cylinder of radius a is filled with viscous liquid, which is initially at rest, and made to rotate with uniform angular velocity ω about its axis. Prove that the velocity of the liquid at time t is

$$2\omega\Sigma \frac{e^{-\lambda^2\nu t} J_1(\lambda r)}{\lambda J_1'(\lambda a)} + \omega r,$$

where the different values of λ are the roots of the equation $J_1(\lambda a) = 0$.

Show also that if the cylinder were surrounded by viscous liquid the solution of the problem might be obtained from the definite integral

$$\int_0^\infty d\lambda \int_0^a e^{-\lambda^2\nu t} \lambda u \phi(u) J_1(\lambda u) J_1(\lambda r)\, du,$$

by properly determining $\phi(u)$ so as to satisfy the boundary conditions.

39. In two-dimensional motion of a viscous fluid, symmetrical with respect to the axis $r = 0$, a general form of the current function is

$$\psi = A\left(t + \frac{\rho r^2}{4\mu}\right) + \Sigma A_n e^{nt} J_0\left(r\sqrt{\frac{-n\rho}{\mu}}\right),$$

where A_n, n are arbitrary complex quantities. (Cf. p. 116.)

40. A right circular cylindrical cavity whose radius is a is made in an infinite conductor; prove that the frequency p of the electrical oscillations about the distribution of electricity where the surface density is proportional to $\cos s\theta$, is given by the equation

$$J_s'(pa/v) = 0,$$

where v is the velocity of propagation of electromagnetic action through the dielectric inside the cavity.

41. Prove that the current function due to a fine circular vortex, of radius c and strength m, may be expressed in the form

$$mra \int_0^\infty e^{\pm\lambda(z-z')} J_1(\lambda r) J_1(\lambda c)\, d\lambda,$$

the upper or lower sign being taken according as $z - z'$ is negative or positive.

42. A magnetic pole of strength m is placed in front of an iron plate of magnetic permeability μ and thickness c: if m be the origin of rectangular coordinates x, y, and x be perpendicular and y parallel to the plate, show that Ω the potential behind the plate is given by the equation

$$\Omega = m(1-\rho^2) \int_0^\infty \frac{e^{-xt} J_0(yt)\, dt}{1 - \rho^2 e^{-2ct}},$$

where

$$\rho = \frac{\mu - 1}{\mu + 1}.$$

43. A right circular cylinder of radius a containing air, moving forwards with velocity V at right angles to its axis, is suddenly stopped; prove that ψ the velocity potential inside the cylinder at a point distant r from the axis, and where the radius makes an angle θ with the direction in which the cylinder was moving, is given by the equation

$$\psi = -\Sigma V \cos\theta \frac{J_1(\kappa r)}{J_1''(\kappa a)} \cos\kappa a t,$$

where a is the velocity of sound in air, and the summation is taken for all values of κ which satisfy the equation $J_1'(\kappa a) = 0$.

44. Prove that if the opening of the object-glass of the telescope in the diffraction problem considered at p. 178 above be ring-shaped the intensity of illumination produced by a single point-source at any point of the focal plane is proportional to

$$\frac{4}{(1-p^2)^2} \frac{\{J_1(z) - pJ_1(pz)\}^2}{z^2},$$

if $z = 2\pi Rr/\lambda f$, where R is the outer radius, pR the inner radius of the opening, r the distance of the point illuminated from the geometrical image of the source, and f the focal length of the object-glass.

45. Prove that the integral of the expression in the preceding example taken for a line-source involves the evaluation of an integral of the form

$$\int_0^\infty \frac{J_1(ax) J_1(bx)}{x\sqrt{x^2 - \xi^2}}\, dx.$$

46. Show that

$$J_n(ax)\,J_n(bx) = \frac{x^n a^n b^n}{2^n \sqrt{\pi}\,\Pi\,(n-\frac{1}{2})} \int_0^\pi \frac{J_n\left(x\sqrt{a^2+b^2-2ab\cos\phi}\right)}{\sqrt{a^2+b^2-2ab\cos\phi}} \sin^{2n}\phi\,d\phi.$$

47. Hence prove that

$$\int_z^\infty \frac{J_1(ax)\,J_1(bx)}{x\sqrt{x^2-\xi^2}}\,dx = \frac{ab}{\pi z}\int_0^\pi \frac{\sin\left(\xi\sqrt{a^2+b^2-2ab\cos\phi}\right)}{a^2+b^2-2ab\cos\phi}\sin^2\phi\,d\phi.$$

(Struve.)

48. A solid isotropic sphere is strained symmetrically in the radial direction and is then left to perform radial oscillations: show that if u be the strain at distance r from the centre, k and n the bulk and rigidity moduli, and ρ the density, the equation of motion is

$$\frac{\partial^2 u}{dt^2} = \frac{k+\frac{4}{3}n}{\rho}\left(\frac{\partial^2 u}{\partial r^2} + \frac{4}{r}\frac{\partial u}{\partial r}\right),$$

with the surface condition

$$(k+\tfrac{4}{3}n)\frac{\partial u}{\partial r} + 3k\frac{u}{r} = 0.$$

Prove that the complete solution subject to the condition stated is

$$u = \Sigma \frac{1}{\eta_p^{\frac{3}{2}}} J_{\frac{3}{2}}(\eta_p)\{A_p \sin c_p t + A_p' \cos c_p t\}$$

$$+ \Sigma \frac{1}{\eta_p^{-\frac{3}{2}}} J_{-\frac{3}{2}}(\eta_p)\left\{\frac{1}{r^3}(B_p \sin c_p t + B_p' \cos c_p t)\right\},$$

where

$$\eta_p^2 = c_p^2 \frac{\rho}{k+\frac{4}{3}n} r^2,$$

c_p being the pth root of the equation

$$(k+\tfrac{4}{3}n)\,a\frac{\partial}{\partial a}\left\{\frac{1}{\eta^{\frac{3}{2}}} J_{\frac{3}{2}}(\eta)\right\} + 3k\frac{1}{\eta^{\frac{3}{2}}} J_{\frac{3}{2}}(\eta) = 0,$$

$[\eta^2 = c^2\rho a^2/(k+\frac{4}{3}n)]$ which holds at the surface $r=a$ of the sphere.

Show that for the motion specified $B_p = B_p' = 0$; and [using 108, p. 53 above], prove that, if the initial values of ru, $r\dot{u}$ be $\phi(r)$, $\psi(r)$,

$$A_p = \frac{\int_0^a \eta_p^{\frac{3}{2}} J_{\frac{3}{2}}(\eta_p)\,\psi(r)\,dr}{c_p\int_0^a \{J_{\frac{3}{2}}(\eta_p)\}^2 r\,dr}, \quad A_p' = \frac{\int_0^a \eta_p^{\frac{3}{2}} J_{\frac{3}{2}}(\eta_p)\,\phi(r)\,dr}{\int_0^a \{J_{\frac{3}{2}}(\eta_p)\}^2 r\,dr}.$$

49. Obtain the equation of motion of a simple pendulum of variable length in the form

$$l\frac{d^2\theta}{dt^2} + 2\frac{dl}{dt}\frac{d\theta}{dt} + g\sin\theta = 0,$$

and show that if $l = a + bt$, where a and b are constants, the equation of motion for the small oscillations may be written

$$x\frac{d^2u}{dx^2} + u = 0,$$

where

$$u = l\theta, \qquad x = gl/b^2.$$

Solve the equation in u by means of Bessel functions, and prove that when $b/\sqrt{ga}$ is small, we have approximately

$$\theta = \rho\left(1 - \frac{3bt}{4a}\right)\sin\left(\sqrt{\frac{g}{a}}\,t - \omega\right)$$
$$+ \frac{b\rho}{8\sqrt{ga}}\left(1 - \frac{2gt^2}{a}\right)\cos\left(\sqrt{\frac{g}{a}}\,t - \omega\right),$$

ρ and ω being arbitrary constants.

(See Lecornu, *C. R.* Jan. 15, 1894. The problem is suggested by the swaying of a heavy body let down by a crane.)

50. If the functions $C_s^n(\cos\phi)$ are defined by the identity

$$(1 - 2a\cos\phi + a^2)^{-n} = \sum_{s=0}^{s=\infty} C_s^n(\cos\phi)\,a^s$$

prove that

$$\frac{J_n\left(\sqrt{a^2 + b^2 - 2ab\cos\phi}\right)}{(a^2 + b^2 - 2ab\cos\phi)^{\frac{1}{2}n}}$$
$$= 2^n\Pi(n-1)\sum_{s=0}^{s=\infty}(n+s)\frac{J_{n+s}(a)}{a^n}\frac{J_{n+s}(b)}{b^n}C_s^n(\cos\phi).$$

(Gegenbauer.)

51. Prove that if $n > m > -1$

$$\int_0^\infty J_m(bx)\,J_n(ax)\,x^{m-n+1}dx = \frac{b^m}{a^n}\frac{(a^2 - b^2)^{n-m-1}}{2^{n-m-1}\Pi(n-m-1)},$$

if $a > b$; and that the value of the integral is zero if $a < b$.

(Sonine.)

52. If $m > -\frac{1}{2}$

$$\int_0^\infty J_m(ax)\,J_m(bx)\,J_m(cx)\,x^{1-m}dx$$
$$= \frac{[(a+b+c)(a+b-c)(b+c-a)(c+a-b)]^{m-\frac{1}{2}}}{\sqrt{\pi}\,.\,2^{3m-1}\Pi\left(m-\frac{1}{2}\right).\,a^mb^mc^m},$$

provided that $b+c-a$, $c+a-b$, $a+b-c$ are all positive; and that if this is not the case the value of the integral is zero.

(Sonine.)

53. Prove that

$$J_0(r) = \frac{2}{\pi}\int_0^\infty \frac{\sin(u+r)}{u+r} J_0(u)\,du,$$

$$Y_0(r) = -2\int_0^\infty \frac{\cos(u+r)}{u+r} J_0(u)\,du.$$

(Sonine and Hobson.)

54. Verify the following expansions :—

(i) $e^{r\cos\theta} J_0(r\sin\theta) = \sum_0^\infty \frac{r^n}{n!} P_n(\cos\theta),$

(ii) $J_0(r\sin\theta) = \sqrt{\frac{2\pi}{r}} \sum_0^\infty \frac{(2n+\frac{1}{2})(2n)!}{2^{2n+1} n!\, n!} P_{2n}(\cos\theta) J_{2n+\frac{1}{2}}(r),$

(iii) $\frac{J_{\frac{1}{2}}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}}} = \frac{2\sqrt{2}}{\pi r} \sum_0^\infty C_{2s}^1 J_{2s+1}(r),$

with the notation of example 50.

(Hobson, *Proc. L. M. S.* xxv.)

55. Prove that

$$\sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{2}\pi} J_n(r\sin\theta) \sin^{n+1}\theta\, d\theta = \frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}},$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{2}\pi} J_{n-\frac{1}{2}}(r\sin\theta) \sin^{n+\frac{1}{2}}\theta\, d\theta = \frac{J_n(r)}{\sqrt{r}},$$

$$\int_0^\infty e^{-\lambda z}\, Y_0(\lambda\rho)\, d\lambda = \frac{1}{\sqrt{z^2+\rho^2}} \log \frac{z+\sqrt{z^2+\rho^2}}{\rho}.$$

(*ibid.*)

FORMULÆ FOR CALCULATION OF THE ROOTS OF BESSEL FUNCTIONS.

In a paper, shortly to be published, "On the roots of the Bessel and certain related functions," for a MS. copy of which we are indebted to the author, Professor J. McMahon obtains the following important results, the first of which has already been given in part.

(i) The sth root, in order of magnitude, of the equation

$$J_n(x) = 0$$

is
$$x_n^{(s)} = \beta - \frac{m-1}{8\beta} - \frac{4(m-1)(7m-31)}{3(8\beta)^3} - \frac{32(m-1)(83m^2-982m+3779)}{15(8\beta)^5} - \frac{64(m-1)(6949m^3-153855m^2+185743m-6277237)}{105(8\beta)^7} - \dots$$ 5

where
$$\beta = \tfrac{1}{4}\pi(2n+4s-1), \quad m = 4n^2.$$

(ii) The sth root, in order of magnitude, of the equation

$$J_n'(x) = 0$$

is
$$x_n^{(s)} = \gamma - \frac{m+3}{8\gamma} - \frac{4(7m^2+82m-9)}{3(8\gamma)^3} - \frac{32(83m^3+2075m^2-3039m+3537)}{15(8\gamma)^3} - \dots$$

where
$$\gamma = \tfrac{1}{4}\pi(2n+4s+1), \quad m = 4n^2.$$

(iii) The sth root, in order of magnitude, of the equation

$$\frac{d}{dx}\{x^{-\frac{1}{2}} J_n(x)\} = 0$$

is
$$x_n^{(s)} = \gamma - \frac{m+7}{8\gamma} - \frac{4(7m^2+154m+95)}{3(8\gamma)^3} - \frac{32(83m^3+3535m^2+3561m+6133)}{15(8\gamma)^3} - \dots$$

where, as above,
$$\gamma = \tfrac{1}{4}\pi(2n+4s+1), \quad m = 4n^2.$$

(iv) The sth root, in order of magnitude, of the equation

$$Y_n(x) + (\gamma - \log 2) J_n(x) = 0$$

is given by the series for $x_n^{(s)}$ in (i) if $\beta - \frac{1}{2}\pi$ be therein substituted for β.

(v) The sth root, in order of magnitude, of the equation

$$\frac{d}{dx}\{Y_n(x) + (\gamma - \log 2) J_n(x)\} = 0$$

is given by the series for $x_n^{(s)}$ in (ii) if $\gamma - \frac{1}{2}\pi$ be therein substituted for γ. [The γ in the expression here differentiated is of course Euler's constant, and is not to be confounded with the γ in the expression for the root.]

(vi) The sth root, in order of magnitude, of the equation

$$\frac{G_n(x)}{J_n(x)} - \frac{G_n(\rho x)}{J_n(\rho x)} = 0, \quad \rho > 1,$$

where $-G_n(x) = Y_n(x) + (\gamma - \log 2) J_n(x)$, or, which is the same,

$$\frac{Y_n(x)}{J_n(x)} - \frac{Y_n(\rho x)}{J_n(\rho x)} = 0,$$

is

$$x_n^{(s)} = \delta + \frac{p}{\delta} + \frac{q - p^2}{\delta^3} + \frac{r - 4pq + 2p^3}{\delta^5} + \dots\dots,$$

where

$$\delta = \frac{s\pi}{\rho - 1}, \quad p = \frac{m-1}{8\rho}, \quad q = \frac{4(m-1)(m-25)(\rho^3 - 1)}{3(8\rho)^3(\rho - 1)}$$

$$r = \frac{32(m-1)(m^2 - 114m + 1073)(\rho^5 - 1)}{5(8\rho^3)(\rho - 1)}, \quad m = 4n^2.$$

(vii) The sth root, in order of magnitude, of the equation

$$\frac{Y'_n(x)}{J'_n(x)} - \frac{Y'_n(\rho x)}{J'_n(\rho x)} = 0, \quad \rho > 1,$$

is given by the same formula as in (vi), but with

$$p = \frac{m+3}{8\rho}, \quad q = \frac{4(m^2 + 46m - 63)(\rho^3 - 1)}{3(8\rho)^3(\rho - 1)},$$

$$r = \frac{32(m^3 + 185m^2 - 2053m + 1899)(\rho^5 - 1)}{5(8\rho)^5(\rho - 1)}.$$

[Of course here also the G functions may be used instead of the Y functions without altering the equation.]

(viii) The sth root, in order of magnitude, of the equation

$$\frac{\frac{d}{dx}\{x^{-\frac{1}{2}} Y_n(x)\}}{\frac{d}{dx}\{x^{-\frac{1}{2}} J_n(x)\}} - \frac{\frac{d}{dx}\{(\rho x)^{-\frac{1}{2}} Y_n(\rho x)\}}{\frac{d}{dx}\{(\rho x)^{-\frac{1}{2}} J_n(\rho x)\}} = 0, \quad \rho > 1,$$

is also given by the formula in (vi), but with

$$p=\frac{m+7}{8\rho},\ q=\frac{4\,(m^2+70m-199)\,(\rho^3-1)}{3\,(8\rho)^3\,(\rho-1)},$$

$$r=\frac{32\,(m^3+245m^2-3693m+4471)\,(\rho^5-1)}{5\,(8\rho)^5\,(\rho-1)}.$$

[As before the G functions may here replace the Y functions.]

The following notes on these equations may be useful.

1. Examples of the equation in (i) are found in all kinds of physical applications, see pp. 56, 96, 178, 219, and elsewhere above.

When $n=\frac{3}{2}$ the equation is equivalent to

$$\tan x = x,$$

which occurs in many problems (see pp. 113, 191, above). The roots of this equation can therefore be calculated by the formula in (i).

2. The equation of which the roots are given in (ii) is also of great importance for physical applications, for example it gives the wave lengths of the vibrations of a fluid within a right cylindrical envelope. It expresses the condition that there is no motion of the gas across the cylindrical boundary. [See Lord Rayleigh's *Theory of Sound*, Vol. II. pp. 265—269.]

When $n=\frac{1}{2}$, the equation is equivalent to

$$\tan x = 2x,$$

and when $n=\frac{3}{2}$, it is equivalent to

$$\tan x = \frac{3x}{3-2x^2},$$

and other equivalent equations can be obtained by means of the Table on p. 42 above.

3. The roots of the equation given in (iii) are required for the problem of waves in a fluid contained within a rigid spherical envelope. The equation is the expression of the surface condition which the motion must fulfil, and $x=\kappa a$, where a is the radius. The roots therefore give the possible values of κ. (See Lord Rayleigh's *Theory of Sound*, Vol. II., p. 231, *et seq.*)

When $n=\frac{1}{2}$, the equation is equivalent to

$$\tan x = x,$$

given also by the equation in (i) when $n=\frac{3}{2}$. Again when $n=\frac{3}{2}$ the equation is equivalent to

$$\tan x = \frac{2x}{2-x^2},$$

which gives the spherical nodes of a gas vibrating within a spherical envelope.

4. The roots of the equation in (vi) are required for many physical problems, for example the problem of the cooling of a body bounded by two coaxial right cylindrical surfaces, or the vibrations of an annular membrane. (See p. 99 above.) The values of x and ρx are those of κa, κb, where a, b are the internal and external radii. The roots of the equation thus give the possible values of κ for the problem.

5. The roots of the equation in (vii) are required for the determination of the wave lengths of the vibrations of a fluid contained between two coaxial right cylindrical surfaces. It is the proper extension of (iii) for this annular space. As before, x and ρx are the values of κa, κb, where a, b are the internal and external radii.

6. In (viii) the equation given is derived from the conditions which must hold at the internal and external surfaces of a fluid vibrating in the space between two concentric and fixed spherical surfaces. The values of x and ρx are as before those of κa, κb, where a, b are the internal and external radii. The roots thus give the possible values of κ for the problem.

7. If for low values of s the formulæ for the roots are any of them not very convergent, it may be preferable to interpolate the values from Tables of the numerical values of the functions, if these are available.

In conclusion, it may be stated that the ten first roots of $J_0(x) = 0$, as calculated by Dr Meissel and given in the paper referred to below, are

$$\begin{aligned}
k_1 &= 2{\cdot}40482\ 55577\\
k_2 &= 5{\cdot}52007\ 81103\\
k_3 &= 8{\cdot}65372\ 79129\\
k_4 &= 11{\cdot}79153\ 44391\\
k_5 &= 14{\cdot}93091\ 77086\\
k_6 &= 18{\cdot}07106\ 39679\\
k_7 &= 21{\cdot}21163\ 66299\\
k_8 &= 24{\cdot}35247\ 15308\\
k_9 &= 27{\cdot}49347\ 91320\\
k_{10} &= 30{\cdot}63460\ 64684
\end{aligned}$$

while, for larger values of n,

$$k_n = (n - \tfrac{1}{4})\,\pi + h_1\delta - h_2\delta^3 + h_3\delta^5 - h_4\delta^7 + h_5\delta^9 - \dots$$

where
$$\delta = \frac{1}{n - \frac{1}{4}},$$

$$\begin{aligned}
\text{Log}\,h_1 &= 8{\cdot}59976\ 01403\\
\text{Log}\,h_2 &= 7{\cdot}41558\ 08514\\
\text{Log}\,h_3 &= 6{\cdot}90532\ 68488\\
\text{Log}\,h_4 &= \dot{6}{\cdot}78108\ 01829\\
\text{Log}\,h_5 &= 6{\cdot}92939\ 63062
\end{aligned}$$

and $\text{Log}\,h_1$ means the common logarithm of h_1 increased by 10.

EXPLANATION OF THE TABLES.

TABLE I. is a reprint of Dr Meissel's "Tafel der Bessel'schen Functionen I_k^0 und I_k^1," originally published in the Berlin *Abhandlungen* for 1888. We are indebted to Dr Meissel and the Berlin Academy of Sciences for permission to include this table in the present work. The only change that has been made is to write $J_0(x)$ and $J_1(x)$ instead of I_k^0 and I_k^1. Three obvious misprints in the column of arguments have been corrected; and the value of $J_0(1{\cdot}71)$ has been altered from ·3932... to ·3922... in accordance with a communication from Dr Meissel.

Table II. is derived from an unpublished MS. very kindly placed at our disposal by its author, Dr Meissel. It gives, for positive integral values of n and x, all the values of $J_n(x)$, from $x=1$ to $x=24$, which are not less than 10^{-18}. The table may be used, among other purposes, for the calculation of $J_n(x)$ when x is not integral. Thus if x lies between two consecutive integers y, $y+1$ we may put $x = y + h$, and then

$$\begin{aligned} J_n(x) &= J_n(y) + hJ_n'(y) + \frac{h^2}{2!} J_n''(y) + \ldots \\ &= J_n(y) + h\left\{\frac{n}{y} J_n(y) - J_{n+1}(y)\right\} \\ &\quad + \frac{h^2}{2}\left\{\left(\frac{n(n-1)}{y^2} - 1\right) J_n(y) + \frac{1}{y} J_{n+1}(y)\right\} + \ldots \end{aligned}$$

We take this opportunity of referring to two papers on the Bessel functions by Dr Meissel contained in the annual reports on the Ober-Realschule at Kiel for the years 1889—90 and 1891—2. It is there shown, among other things, that, when x is given, there is a special value of n for which the function $J_n(x)$ changes sign for the last time from negative to positive; that the function then increases to its absolute maximum, and then diminishes as n increases, with ever increasing rapidity.

Table III., which is taken from the first of the papers just referred to, gives the first 50 roots of the equation $J_1(x) = 0$, with the corre-

sponding values of $J_0(x)$, which are, of course, maximum or minimum values of $J_0(x)$ according as they are positive or negative.

Tables IV., V., and VI. are extracted from the Reports of the British Association for the years 1889 and 1893. The Association table corresponding to V. was thought too long to reprint, so the tabular difference has been taken to be ·01 instead of ·001. These tables do not require any special explanation: the functions I_n are the same as those denoted by that symbol in the present work.

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
0·00	1·000000000000	0·000000000000	0·40	0·960398226660	−0·196026577955
0·01	0·999975000156	−0·004999937500	0·41	0·958414468885	−0·200722502946
0·02	0·999900002500	−0·009999500008	0·42	0·956383826663	−0·205403409375
0·03	0·999775012656	−0·014998312563	0·43	0·954306451921	−0·210068948818
0·04	0·999600039998	−0·019996000267	0·44	0·952182500067	−0·214718774133
0·05	0·999375097649	−0·024992188314	0·45	0·950012129972	−0·219352539483
0·06	0·999100202480	−0·029986502025	0·46	0·947795503959	−0·223969900370
0·07	0·998775375105	−0·034978566876	0·47	0·945532787790	−0·228570513659
0·08	0·998400639886	−0·039968008532	0·48	0·943224150650	−0·233154037611
0·09	0·997976024926	−0·044954452875	0·49	0·940869765137	−0·237720131905
0·10	0·997501562066	−0·049937526036	0·50	0·938469807241	−0·242268457675
0·11	0·996977286887	−0·054916854430	0·51	0·936024456336	−0·246798677529
0·12	0·996403238704	−0·059892064781	0·52	0·933533895163	−0·251310455583
0·13	0·995779460562	−0·064862784157	0·53	0·930998309812	−0·255803457487
0·14	0·995105999233	−0·069828640001	0·54	0·928417889710	−0·260277350453
0·15	0·994382905214	−0·074789260161	0·55	0·925792827604	−0·264731803281
0·16	0·993610232721	−0·079744272921	0·56	0·923123319544	−0·269166486388
0·17	0·992788039685	−0·084693307032	0·57	0·920409564868	−0·273581071836
0·18	0·991916387745	−0·089635991743	0·58	0·917651766187	−0·277975233357
0·19	0·990995342249	−0·094571956833	0·59	0·914850129363	−0·282348646381
0·20	0·990024972240	−0·099500832639	0·60	0·912004863497	−0·286700988064
0·21	0·989005350457	−0·104422250091	0·61	0·909116180910	−0·291031937312
0·22	0·987936553327	−0·109335840739	0·62	0·905184297124	−0·295341174811
0·23	0·986818660958	−0·114241236785	0·63	0·903209430845	−0·299628383050
0·24	0·985651757131	−0·119138071113	0·64	0·900191803946	−0·303893246349
0·25	0·984435929296	−0·124025977323	0·65	0·897131641447	−0·308135450885
0·26	0·983171268563	−0·128904589754	0·66	0·894029171498	−0·312354684718
0·27	0·981857869696	−0·133773543525	0·67	0·890884625356	−0·316550637815
0·28	0·980495831102	−0·138632474553	0·68	0·887698237371	−0·320723002080
0·29	0·979085254825	−0·143481019596	0·69	0·884470244964	−0·324871471373
0·30	0·977626246538	−0·148318816273	0·70	0·881200888607	−0·328995741540
0·31	0·976118915533	−0·153145503099	0·71	0·877890411804	−0·333095510438
0·32	0·974563374711	−0·157960719516	0·72	0·874539061070	−0·337170477956
0·33	0·972959740576	−0·162764105918	0·73	0·871147085910	−0·341220346045
0·34	0·971308133222	−0·167555303687	0·74	0·867714738801	−0·345244818737
0·35	0·969608676323	−0·172333955219	0·75	0·864242275167	−0·349243602175
0·36	0·967861497127	−0·177099703954	0·76	0·860729953361	−0·353216404632
0·37	0·966066726439	−0·181852194406	0·77	0·857178034643	−0·357162936538
0·38	0·964224498614	−0·186591072196	0·78	0·853586783157	−0·361082910503
0·39	0·962334951548	−0·191315984074	0·79	0·849956465910	−0·364976041342
0·40	0·960398226660	−0·196026577955	0·80	0·846287352750	−0·368842046094

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
0·80	0·846287352750	−0·368842046094	1·20	0·671132744264	−0·498289057567
0·81	0·842579716344	−0·372680644052	1·21	0·666137120084	−0·500829672641
0·82	0·838833832154	−0·376491556779	1·22	0·661116273214	−0·503333567025
0·83	0·835049978414	−0·380274508136	1·23	0·656070571706	−0·505800572628
0·84	0·831228436109	−0·384029224303	1·24	0·651000385275	−0·508230524394
0·85	0·827369488950	−0·387755433798	1·25	0·645906085271	−0·510623260320
0·86	0·823473423352	−0·391452867506	1·26	0·640788044651	−0·512978621467
0·87	0·819540528409	−0·395121258696	1·27	0·635646637944	−0·515296451971
0·88	0·815571095868	−0·398760343044	1·28	0·630482241224	−0·517576599061
0·89	0·811565420110	−0·402369858653	1·29	0·625295232074	−0·519818913063
0·90	0·807523798123	−0·405949546079	1·30	0·620085989562	−0·522023247415
0·91	0·803446529473	−0·409499148347	1·31	0·614854894203	−0·524189458680
0·92	0·799333916288	−0·413018410976	1·32	0·609602327933	−0·526317406556
0·93	0·795186263226	−0·416507081996	1·33	0·604328674074	−0·528406953885
0·94	0·791003877452	−0·419964911971	1·34	0·599034317304	−0·530457966666
0·95	0·786787068613	−0·423391654020	1·35	0·593719643626	−0·532470314063
0·96	0·782536148813	−0·426787063833	1·36	0·588385040333	−0·534443868418
0·97	0·778251432583	−0·430150899695	1·37	0·583030895983	−0·536378505258
0·98	0·773933236862	−0·433482922506	1·38	0·577657600358	−0·538274103303
0·99	0·769581880965	−0·436782895795	1·39	0·572265544440	−0·540130544481
1·00	0·765197686558	−0·440050585745	1·40	0·566855120374	−0·541947713931
1·01	0·760780977632	−0·443285761209	1·41	0·561426721439	−0·543725500014
1·02	0·756332080477	−0·446488193730	1·42	0·555980742014	−0·545463794323
1·03	0·751851323654	−0·449657657556	1·43	0·550517577543	−0·547162491686
1·04	0·747339037965	−0·452793929666	1·44	0·545037624510	−0·548821490179
1·05	0·742795556434	−0·455896789778	1·45	0·539541280398	−0·550440691132
1·06	0·738221214269	−0·458966020374	1·46	0·534028943664	−0·552019999133
1·07	0·733616348841	−0·462001406715	1·47	0·528501013700	−0·553559322039
1·08	0·728981299655	−0·465002736858	1·48	0·522957890804	−0·555058570983
1·09	0·724316408322	−0·467969801675	1·49	0·517399976146	−0·556517660374
1·10	0·719622018528	−0·470902394866	1·50	0·511827671736	−0·557936507910
1·11	0·714898476008	−0·473800312980	1·51	0·506241380391	−0·559315034582
1·12	0·710146128520	−0·476663355426	1·52	0·500641505700	−0·560653164677
1·13	0·705365325811	−0·479491324496	1·53	0·495028451994	−0·561950825786
1·14	0·700556419592	−0·482284025373	1·54	0·489402624312	−0·563207948806
1·15	0·695719763505	−0·485041266154	1·55	0·483764428365	−0·564424467949
1·16	0·690855713099	−0·487762857858	1·56	0·478114270507	−0·565600320742
1·17	0·685964625798	−0·490448614448	1·57	0·472452557702	−0·566735448033
1·18	0·681046860871	−0·493098352841	1·58	0·466779697485	−0·567829793994
1·19	0·676102779403	−0·495711892924	1·59	0·461096097935	−0·568883306126
1·20	0·671132744264	−0·498289057567	1·60	0·455402167639	−0·569895935262

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
1·60	0·455402167639	−0·569895935262	2·00	0·223890779141	−0·576724807757
1·61	0·449698315660	−0·570867635566	2·01	0·218126821326	−0·576060090955
1·62	0·443984951500	−0·571798364542	2·02	0·212369710458	−0·575355433450
1·63	0·438262485071	−0·572688083032	2·03	0·206619845483	−0·574610928248
1·64	0·432531326660	−0·573536755217	2·04	0·200877624399	−0·573826671543
1·65	0·426791886896	−0·574344348624	2·05	0·195143444226	−0·573002762707
1·66	0·421044576715	−0·575110834122	2·06	0·189417700977	−0·572139304279
1·67	0·415289807326	−0·575836185927	2·07	0·183700789621	−0·571236401957
1·68	0·409527990183	−0·576520381599	2·08	0·177993104055	−0·570294164587
1·69	0·403759536945	−0·577163402048	2·09	0·172295037073	−0·569312704151
1·70	0·397984859446	−0·577765231529	2·10	0·166606980332	−0·568292135757
1·71	0·392204369660	−0·578325857645	2·11	0·160929324324	−0·567232577628
1·72	0·386418479668	−0·578845271345	2·12	0·155262458341	−0·566134151091
1·73	0·380627601627	−0·579323466925	2·13	0·149606770449	−0·564996980564
1·74	0·374832147732	−0·579760442028	2·14	0·143962647452	−0·563821193544
1·75	0·369032530185	−0·580156197639	2·15	0·138330474865	−0·562606920596
1·76	0·363229161163	−0·580510738087	2·16	0·132710636881	−0·561354295339
1·77	0·357422452782	−0·580824071043	2·17	0·127103516344	−0·560063454436
1·78	0·351612817064	−0·581096207515	2·18	0·121509494713	−0·558734537577
1·79	0·345800665906	−0·581327161851	2·19	0·115928952037	−0·557367687469
1·80	0·339986411043	−0·581516951731	2·20	0·110362266922	−0·555963049819
1·81	0·334170464016	−0·581665598167	2·21	0·104809816503	−0·554520773326
1·82	0·328353236143	−0·581773125501	2·22	0·099271976413	−0·553041009659
1·83	0·322535138478	−0·581839561397	2·23	0·093749120752	−0·551523913451
1·84	0·316716581784	−0·581864936842	2·24	0·088241622061	−0·549969642278
1·85	0·310897976496	−0·581849286141	2·25	0·082749851289	−0·548378356647
1·86	0·305079732690	−0·581792646910	2·26	0·077274177765	−0·546750219981
1·87	0·299262260050	−0·581695060074	2·27	0·071814969172	−0·545085398603
1·88	0·293445967833	−0·581556569863	2·28	0·066372591512	−0·543384061721
1·89	0·286631264839	−0·581377223803	2·29	0·060947409082	−0·541646381412
1·90	0·281818559374	−0·581157072713	2·30	0·055539784446	−0·539872532604
1·91	0·276008259222	−0·580896170703	2·31	0·050150078400	−0·538062693065
1·92	0·270200771606	−0·580594575158	3·32	0·044778649952	−0·536217043381
1·93	0·264396503162	−0·580252346743	2·33	0·039425856288	−0·534335766941
1·94	0·258595859901	−0·579869549389	2·34	0·034092052749	−0·532419049921
1·95	0·252799247180	−0·579446250290	2·35	0·028777592796	−0·530467081267
1·96	0·247007069667	−0·578982519892	2·36	0·023482827990	−0·528480052675
1·97	0·241219731308	−0·578478431892	2·37	0·018208107961	−0·526458158577
1·98	0·235437635298	−0·577934063221	2·38	0·012953780380	−0·524401596119
1·99	0·229661184046	−0·577349494047	2·39	0·007720190934	−0·522310565146
2·00	0·223890779141	−0·576724807757	2·40	0·002507683297	−0·520185268182

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
2·40	+0·002507683297	−0·520185268182	2·80	−0·185036033364	−0·409709246852
2·41	−0·002683400894	−0·518025910413	2·81	−0·189116518066	−0·406383733066
2·42	−0·007852722067	−0·515832699667	2·82	−0·193163629309	−0·403034604450
2·43	−0·012999942745	−0·513605846395	2·83	−0·197177132431	−0·399662158463
2·44	−0·018124727564	−0·511345563651	2·84	−0·201156795751	−0·396266694238
2·45	−0·023226743305	−0·509052067073	2·85	−0·205102390590	−0·392848512558
2·46	−0·028305658919	−0·506725574866	2·86	−0·209013691285	−0·389407915829
2·47	−0·033361145552	−0·504366307779	2·87	−0·212890475203	−0·385945208051
2·48	−0·038392876569	−0·501974489084	2·88	−0·216732522761	−0·382460694795
2·49	−0·043400527581	−0·499550344558	2·89	−0·220539617438	−0·378954683174
2·50	−0·048383776468	−0·497094102464	2·90	−0·224311545792	−0·375427481813
2·51	−0·053342303407	−0·494605993526	2·91	−0·228048097475	−0·371879400828
2·52	−0·058275790893	−0·492086250909	2·92	−0·231749065248	−0·368310751792
2·53	−0·063183923765	−0·489535110203	2·93	−0·235414244994	−0·364721847712
2·54	−0·068066389230	−0·486952809393	2·94	−0·239043435734	−0·361113003001
2·55	−0·072922876886	−0·484339588844	2·95	−0·242636439638	−0·357484533446
2·56	−0·077753078750	−0·481695691279	2·96	−0·246193062043	−0·353836756187
2·57	−0·082556689272	−0·479021361753	2·97	−0·249713111464	−0·350169989683
2·58	−0·087333405369	−0·476316847635	2·98	−0·253196399605	−0·346484553686
2·59	−0·092082926441	−0·473582398581	2·99	−0·256642741376	−0·342780769216
2·60	−0·096804954397	−0·470818266518	3·00	−0·260051954902	−0·339058958526
2·61	−0·101499193675	−0·468024705615	3·01	−0·263423861537	−0·335319445081
2·62	−0·106165351268	−0·465201972264	3·02	−0·266758285876	−0·331562553524
2·63	−0·110803136741	−0·462350325057	3·03	−0·270055055766	−0·327788609651
2·64	−0·115412262258	−0·459470024758	3·04	−0·273314002318	−0·323997940380
2·65	−0·119992442602	−0·456561334286	3·05	−0·276534959916	−0·320190873724
2·66	−0·124543395193	−0·453624518688	3·06	−0·279717766231	−0·316367738762
2·67	−0·129064840115	−0·450659845115	3·07	−0·282862262330	−0·312528865609
2·68	−0·133556500133	−0·447667582797	3·08	−0·285968292186	−0·308674585389
2·69	−0·138018100713	−0·444648003025	3·09	−0·289035703688	−0·304805230202
2·70	−0·142449370046	−0·441601379118	3·10	−0·292064347651	−0·300921133101
2·71	−0·146850039066	−0·438527986406	3·11	−0·295054078324	−0·297022628058
2·72	−0·151219841469	−0·435428102199	3·12	−0·298004753302	−0·293110049938
2·73	−0·155558513735	−0·432302005768	3·13	−0·300916233531	−0·289183734465
2·74	−0·159865795147	−0·429149978317	3·14	−0·303788383321	−0·285244018200
2·75	−0·164141427809	−0·425972302958	3·15	−0·306621070350	−0·281291238504
2·76	−0·168385156663	−0·422769264686	3·16	−0·309414165674	−0·277325733514
2·77	−0·172596729515	−0·419541150353	3·17	−0·312167543732	−0·273347842110
2·78	−0·176775897046	−0·416288248646	3·18	−0·314881082360	−0·269357903890
2·79	−0·180922412832	−0·413010850055	3·19	−0·317554662788	−0·265356259134
2·80	−0·185036033364	−0·409709246852	3·20	−0·320188169657	−0·261343248781

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
3·20	-0·320188169657	-0·261343248781	3·60	-0·391768983701	-0·095465547178
3·21	-0·322781491017	-0·257319214392	3·61	-0·392702729637	-0·091284136789
3·22	-0·325334518339	-0·253284498129	3·62	-0·393594676939	-0·087105877039
3·23	-0·327847146516	-0·249239442719	3·63	-0·394444858817	-0·082931108843
3·24	-0·330319273873	-0·245184391424	3·64	-0·395253311888	-0·078760172463
3·25	-0.332750802171	-0·241119688015	3·65	-0·396020076171	-0·074593407483
3·26	-0·335141636607	-0·237045676741	3·66	-0·396745195072	-0·070431152776
3·27	-0·337491685828	-0·232962702298	3·67	-0·397428715388	-0·066273746480
3·28	-0·339800861926	-0·228871109797	3·68	-0·398070687288	-0·062121525964
3·29	-0·342069080449	-0·224771244740	3·69	-0·398671164315	-0·057974827802
3·30	-0·344296260399	-0·220663452985	3·70	-0·399230203371	-0·053833987745
3·31	-0·346482324240	-0·216548080719	3·71	-0·399747864713	-0·049699340694
3·32	-0·348627197900	-0·212425474424	3·72	-0·400224211942	-0·045571220667
3·33	-0·350730810771	-0·208295980854	3·73	-0·400659311994	-0·041449960775
3·34	-0·352793095716	-0·204159946997	3·74	-0·401053235132	-0·037335893193
3·35	-0·354813989067	-0·200017720051	3·75	-0·401406054936	-0·033229349130
3·36	-0·356793430631	-0·195869647392	3·76	-0·401717848294	-0·029130658803
3·37	-0·358731363688	-0·191716076543	3·77	-0·401988695389	-0·025040151411
3·38	-0·360627734994	-0·187557355145	3·78	-0·402218679692	-0·020958155102
3·39	-0·362482494781	-0·183393830929	3·79	-0·402407887951	-0·016884996950
3·40	-0·364295596762	-0·179225851682	3·80	-0·402556410179	-0·012821002927
3·41	-0·366066998124	-0·175053765218	3·81	-0·402664339640	-0·008766497873
3·42	-0·367796659535	-0·170877919353	3·82	-0·402731772845	-0·004721805471
3·43	-0·369484545139	-0·166698661869	3·83	-0·402758809533	-0·000687248221
3·44	-0·371130622559	-0·162516340485	3·84	-0·402745552664	+0·003336852592
3·45	-0·372734862895	-0·158331302831	3·85	-0·402692108403	+0·007350176918
3·46	-0·374297240720	-0·154143896414	3·86	-0·402598586110	+0·011352405975
3·47	-0·375817734085	-0·149954468592	3·87	-0·402465098327	+0·015343222272
3·48	-0·377296324511	-0·145763366540	3·88	-0·402291760761	+0·019322309635
3·49	-0·378732996992	-0·141570937221	3·89	-0·402078692280	+0·023289353237
3·50	-0·380127739987	-0·137377527362	3·90	-0·401826014888	+0·027244039621
3·51	-0·381480545425	-0·133183483416	3·91	-0·401533853719	+0·031186056727
3·52	-0·382791408696	-0·128989151538	3·92	-0·401202337020	+0·035115093918
3·53	-0·384060328649	-0·124794877553	3·93	-0·400831596137	+0·039030842006
3·54	-0·385287307591	-0·120601006927	3·94	-0·400421765502	+0·042932993278
3·55	-0·386472351282	-0·116407884739	3·95	-0·399972982615	+0·046821241521
3·56	-0·387615468930	-0·112215855647	3·96	-0·399485388031	+0·050695282047
3·57	-0·388716673186	-0·108025263865	3·97	-0·398959125344	+0·054554811719
3·58	-0·389775980144	-0·103836453128	3·98	-0·398394341172	+0·058399528975
3·59	-0·390793409330	-0·099649766668	3·99	-0·397791185139	+0·062229133855
3·60	-0·391768983701	-0·095465547178	4·00	-0·397149809864	+0·066043328024

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
4·00	−0·397149809864	+0·066043328024	4·40	−0·342256790004	+0·202775521923
4·01	−0·396470370937	+0·069841814795	4·41	−0·340214269569	+0·205724220583
4·02	−0·395753026909	+0·073624299158	4·42	−0·338142392830	+0·208646748043
4·03	−0·394997939273	+0·077390487802	4·43	−0·336041422538	+0·211542896739
4·04	−0·394205272445	+0·081140089137	4·44	−0·333911623508	+0·214412461634
4·05	−0·393375193748	+0·084872813321	4·45	−0·331753262593	+0·217255240239
4·06	−0·392507873396	+0·088588372282	4·46	−0·329566608658	+0·220071032626
4·07	−0·391603484474	+0·092286479742	4·47	−0·327351932553	+0·222859641442
4·08	−0·390662202921	+0·095966851242	4·48	−0·325109507090	+0·225620871929
4·09	−0·389684207511	+0·099629204162	4·49	−0·322839607016	+0·228354531934
4·10	−0·388669679836	+0·103273257747	4·50	−0·320542508985	+0·231060431923
4·11	−0·387618804284	+0·106898733130	4·51	−0·318218491534	+0·233738385002
4·12	−0·386531768024	+0·110505353352	4·52	−0·315867835056	+0·236388206923
4·13	−0·385408760984	+0·114092843385	4·53	−0·313490821772	+0·239009716103
4·14	−0·384249975834	+0·117660930159	4·54	−0·311087735706	+0·241602733636
4·15	−0·383055607963	+0·121209342578	4·55	−0·308658862659	+0·244167083306
4·16	−0·381825855461	+0·124737811545	4·56	−0·306204490179	+0·246702591599
4·17	−0·380560919100	+0·128246069984	4·57	−0·303724907535	+0·249209087721
4·18	−0·379261002313	+0·131733852860	4·58	−0·301220405692	+0·251686403603
4·19	−0·377926311172	+0·135200897203	4·59	−0·298691277281	+0·254134373919
4·20	−0·376557054368	+0·138646942126	4·60	−0·296137816574	+0·256552836097
4·21	−0·375153443190	+0·142071728849	4·61	−0·293560319453	+0·258941630330
4·22	−0·373715691507	+0·145475000717	4·62	−0·290959083385	+0·261300599586
4·23	−0·372244015741	+0·148856503224	4·63	−0·288334407392	+0·263629589622
4·24	−0·370738634848	+0·152215984028	4·64	−0·285686592028	+0·265928448996
4·25	−0·369199770300	+0·155553192978	4·65	−0·283015939344	+0·268197029073
4·26	−0·367627646055	+0·158867882130	4·66	−0·280322752864	+0·270435184041
4·27	−0·366022488543	+0·162159805765	4·67	−0·277607337557	+0·272642770917
4·28	−0·364384526637	+0·165428720414	4·68	−0·274869999807	+0·274819649559
4·29	−0·362713991635	+0·168674384873	4·69	−0·272111047384	+0·276965682678
4·30	−0·361011117237	+0·171896560222	4·70	−0·269330789420	+0·279080735843
4·31	−0·359276139517	+0·175095009847	4·71	−0·266529536373	+0·281164677493
4·32	−0·357509296907	+0·178269499458	4·72	−0·263707600004	+0·283217378945
4·33	−0·355710830168	+0·181419797104	4·73	−0·260865293347	+0·285238714404
4·34	−0·353880982370	+0·184545673196	4·74	−0·258002930679	+0·287228560970
4·35	−0·352019998867	+0·187646900522	4·75	−0·255120827491	+0·289186798647
4·36	−0·350128127272	+0·190723254265	4·76	−0·252219300460	+0·291113310352
4·37	−0·348205617435	+0·193774512024	4·77	−0·249298667418	+0·293007981919
4·38	−0·346252721418	+0·196800453825	4·78	−0·246359247327	+0·294870702112
4·39	−0·344269693470	+0·199800862145	4·79	−0·243401360242	+0·296701362626
4·40	−0·342256790004	+0·202775521923	4·80	−0·240425327291	+0·298499858100

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
4·80	−0·240425327291	+0·298499858100	5·20	−0·110290439791	+0·343223005872
4·81	−0·237431470639	+0·300266086117	5·21	−0·106856051931	+0·343648917051
4·82	−0·234420113459	+0·301999947217	5·22	−0·103417574396	+0·344040944641
4·83	−0·231391579906	+0·303701344899	5·23	−0·099975345904	+0·344399112424
4·84	−0·228346195084	+0·305370185627	5·24	−0·096529704924	+0·344723447160
4·85	−0·225284285019	+0·307006378837	5·25	−0·093080989639	+0·345013978579
4·86	−0·222206176625	+0·308609836942	5·26	−0·089629537922	+0·345270739379
4·87	−0·219112197679	+0·310180475336	5·27	−0·086175687302	+0·345493765217
4·88	−0·216002676790	+0·311718212399	5·28	−0·082719774939	+0·345683094703
4·89	−0·212877943365	+0·313222969504	5·29	−0·079262137591	+0·345838769398
4·90	−0·209738327585	+0·314694671015	5·30	−0·075803111586	+0·345960833801
4·91	−0·206584160372	+0·316133244299	5·31	−0·072343032791	+0·346049335349
4·92	−0·203415773359	+0·317538619723	5·32	−0·068882236587	+0·346104324405
4·93	−0·200233498860	+0·318910730662	5·33	−0·065421057834	+0·346125854251
4·94	−0·197037669840	+0·320249513497	5·34	−0·061959830846	+0·346113981085
4·95	−0·193828619886	+0·321554907624	5·35	−0·058498889359	+0·346068764007
4·96	−0·190606683176	+0·322826855452	5·36	−0·055038566506	+0·345990265014
4·97	−0·187372194447	+0·324065302408	5·37	−0·051579194783	+0·345878548995
4·98	−0·184125488969	+0·325270196936	5·38	−0·048121106024	+0·345733683714
4·99	−0·180866902512	+0·326441490501	5·39	−0·044664631371	+0·345555739809
5·00	−0·177596771314	+0·327579137591	5·40	−0·041210101245	+0·345344790780
5·01	−0·174315432057	+0·328683095718	5·41	−0·037757845318	+0·345100912978
5·02	−0·171023221828	+0·329753325415	5·42	−0·034308192484	+0·344824185600
5·03	−0·167720478098	+0·330789790243	5·43	−0·030861470832	+0·344514690673
5·04	−0·164407538685	+0·331792456787	5·44	−0·027418007614	+0·344172513049
5·05	−0·161084741725	+0·332761294658	5·45	−0·023978129221	+0·343797740393
5·06	−0·157752425645	+0·333696276491	5·46	−0·020542161155	+0·343390463171
5·07	−0·154410929130	+0·334597377947	5·47	−0·017110427996	+0·342950774642
5·08	−0·151060591092	+0·335464577712	5·48	−0·013683253380	+0·342478770844
5·09	−0·147701750643	+0·336297857492	5·49	−0·010260959967	+0·341974550584
5·10	−0·144334747061	+0·337097202018	5·50	−0·006843869418	+0·341438215429
5·11	−0·140959919761	+0·337862599041	5·51	−0·003432302361	+0·340869869689
5·12	−0·137577608269	+0·338594039331	5·52	−0·000026578369	+0·340269620408
5·13	−0·134188152185	+0·339291516672	5·53	+0·003372984068	+0·339637577354
5·14	−0·130791891157	+0·339955027866	5·54	+0·006766067573	+0·338973853000
5·15	−0·127389164849	+0·340584572725	5·55	+0·010152355907	+0·338278562520
5·16	−0·123980312914	+0·341180154069	5·56	+0·013531533995	+0·337551823766
5·17	−0·120565674960	+0·341741777728	5·57	+0·016903287956	+0·336793757265
5·18	−0·117145590523	+0·342269452530	5·58	+0·020267305125	+0·336004486197
5·19	−0·113720399033	+0·342763190303	5·59	+0·023623274084	+0·335184136388
5·20	−0·110290439791	+0·343223005872	5·60	+0·026970884685	+0·334332836291

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
5·60	+0·026970884685	+0·334332836291	6·00	+0·150645257251	+0·276683858128
5·61	+0·030309828079	+0·333450716975	6·01	+0·153402218596	+0·274704492725
5·62	+0·033639796739	+0·332537912108	6·02	+0·156139269116	+0·272701730538
5·63	+0·036960484490	+0·331594557948	6·03	+0·158856175969	+0·270675796964
5·64	+0·040271586530	+0·330620793320	6·04	+0·161552708575	+0·268626919220
5·65	+0·043572799459	+0·329616759609	6·05	+0·164228638636	+0·266555326316
5·66	+0·046863821304	+0·328582600738	6·06	+0·166883740153	+0·264461249036
5·67	+0·050144351544	+0·327518463159	6·07	+0·169517789443	+0·262344919911
5·68	+0·053414091135	+0·326424495830	6·08	+0·172130565159	+0·260206573201
5·69	+0·056672742533	+0·325300850207	6·09	+0·174721848302	+0·258046444869
5·70	+0·059920009724	+0·324147680223	6·10	+0·177291422243	+0·255864772558
5·71	+0·063155598244	+0·322965142271	6·11	+0·179839072737	+0·253661795571
5·72	+0·066379215205	+0·321753395193	6·12	+0·182364587942	+0·251437754842
5·73	+0·069590569321	+0·320512600255	6·13	+0·184867758430	+0·249192892918
5·74	+0·072789370930	+0·319242921139	6·14	+0·187348377209	+0·246927453930
5·75	+0·075975332017	+0·317944523919	6·15	+0·189806239737	+0·244641683576
5·76	+0·079148166242	+0·316617577048	6·16	+0·192241143934	+0·242335829091
5·77	+0·082307588961	+0·315262251336	6·17	+0·194652890201	+0·240010139225
5·78	+0·085453317250	+0·313878719939	6·18	+0·197041281434	+0·237664864220
5·79	+0·088585069926	+0·312467158333	6·19	+0·199406123040	+0·235300255786
5·80	+0·091702567575	+0·311027744304	6·20	+0·201747222949	+0·232916567073
5·81	+0·094805532571	+0·309560657922	6·21	+0·204064391629	+0·230514052652
5·82	+0·097893689100	+0·308066081529	6·22	+0·206357442103	+0·228092968487
5·83	+0·100966763183	+0·306544199716	6·23	+0·208626189957	+0·225653571908
5·84	+0·104024482698	+0·304995199305	6·24	+0·210870453362	+0·223196121594
5·85	+0·107066577404	+0·303419269333	6·25	+0·213090053077	+0·220720877539
5·86	+0·110092778957	+0·301816601028	6·26	+0·215284812471	+0·218228101034
5·87	+0·113102820941	+0·300187387793	6·27	+0·217454557531	+0·215718054638
5·88	+0·116096438881	+0·298531825185	6·28	+0·219599116876	+0·213191002155
5·89	+0·119073370272	+0·296850110895	6·29	+0·221718321770	+0·210647208606
5·90	+0·122033354593	+0·295142444729	6·30	+0·223812006132	+0·208086940207
5·91	+0·124976133333	+0·293409028587	6·31	+0·225880006549	+0·205510464342
5·92	+0·127901450011	+0·291650066443	6·32	+0·227922162289	+0·202918049537
5·93	+0·130809050195	+0·289865764324	6·33	+0·229938315309	+0·200309965435
5·94	+0·133698681524	+0·288056330291	6·34	+0·231928310269	+0·197686482769
5·95	+0·136570093728	+0·286221974417	6·35	+0·233891994542	+0·195047873339
5·96	+0·139423038646	+0·284362908764	6·36	+0·235829218223	+0·192394409984
5·97	+0·142257270250	+0·282479347366	6·37	+0·237739834141	+0·189726366557
5·98	+0·145072544661	+0·280571506204	6·38	+0·239623697870	+0·187044017898
5·99	+0·147868620168	+0·278639603186	6·39	+0·241480667734	+0·184347639808
6·00	+0·150645257251	+0·276683858128	6·40	+0·243310604823	+0·181637509024

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
6·40	+0·243310604823	+0·181637509024	6·80	+0·293095603104	+0·065218663402
6·41	+0·245113372998	+0·178913903193	6·81	+0·293732652315	+0·062190881458
6·42	+0·246888838899	+0·176177100845	6·82	+0·294339415275	+0·059161461866
6·43	+0·248636871957	+0·173427381364	6·83	+0·294915877066	+0·056130696324
6·44	+0·250357344403	+0·170665024967	6·84	+0·295462025686	+0·053098876291
6·45	+0·252050131270	+0·167890312675	6·85	+0·295977852047	+0·050066292954
6·46	+0·253715110409	+0·165103526284	6·86	+0·296463349971	+0·047033237205
6·47	+0·255352162491	+0·162304948344	6·87	+0·296918516185	+0·043999999614
6·48	+0·256961171015	+0·159494862126	6·88	+0·297343350324	+0·040966870403
6·49	+0·258542022319	+0·156673551601	6·89	+0·297737854921	+0·037934139418
6·50	+0·260094605582	+0·153841301410	6·90	+0·298102035405	+0·034902096105
6·51	+0·261618812832	+0·150998396839	6·91	+0·298435900099	+0·031871029480
6·52	+0·263114538957	+0·148145123790	6·92	+0·298739460212	+0·028841228107
6·53	+0·264581681702	+0·145281768758	6·93	+0·299012729839	+0·025812980070
6·54	+0·266020141682	+0·142408618801	6·94	+0·299255725950	+0·022786572947
6·55	+0·267429822386	+0·139525961513	6·95	+0·299468468391	+0·019762293785
6·56	+0·268810630181	+0·136634085000	6·96	+0·299650979874	+0·016740429070
6·57	+0·270162474318	+0·133733277851	6·97	+0·299803285973	+0·013721264707
6·58	+0·271485266933	+0·130823829111	6·98	+0·299925415120	+0·010705085992
6·59	+0·272778923059	+0·127906028255	6·99	+0·300017398594	+0·007692177584
6·60	+0·274043360624	+0·124980165161	7·00	+0·300079270520	+0·004682823482
6·61	+0·275278500456	+0·122046530081	7·01	+0·300111067856	+0·001677306999
6·62	+0·276484266288	+0·119105413617	7·02	+0·300112830394	−0·001324089265
6·63	+0·277660584760	+0·116157106694	7·03	+0·300084600744	−0·004321083446
6·64	+0·278807385424	+0·113201900529	7·04	+0·300026424335	−0·007313394442
6·65	+0·279924600745	+0·110240086609	7·05	+0·299938349401	−0·010300741939
6·66	+0·281012166103	+0·107271956661	7·06	+0·299820426973	−0·013282846438
6·67	+0·282070019798	+0·104297802626	7·07	+0·299672710878	−0·016259429273
6·68	+0·283098103049	+0·101317916630	7·08	+0·299495257720	−0·019230212645
6·69	+0·284096359998	+0·098332590962	7·09	+0·299288126879	−0·022194919639
6·70	+0·285064737711	+0·095342118041	7·10	+0·299051380502	−0·025153274254
6·71	+0·286003186176	+0·092346790394	7·11	+0·298785083486	−0·028105001425
6·72	+0·286911658311	+0·089346900625	7·12	+0·298489303478	−0·031049827049
6·73	+0·287790109957	+0·086342741391	7·13	+0·298164110861	−0·033987478007
6·74	+0·288638499883	+0·083334605375	7·14	+0·297809578741	−0·036917682190
6·75	+0·289456789785	+0·080322785255	7·15	+0·297425782943	−0·039840168524
6·76	+0·290244944284	+0·077307573684	7·16	+0·297012801997	−0·042754666991
6·77	+0·291002930929	+0·074289263257	7·17	+0·296570717126	−0·045660908657
6·78	+0·291730720194	+0·071268146488	7·18	+0·296099612239	−0·048558625692
6·79	+0·292428285479	+0·068244515780	7·19	+0·295599573917	−0·051447551397
6·80	+0·293095603104	+0·065218663402	7·20	+0·295070691401	−0·054327420222

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
7·20	+0·295070691401	−0·054327420222	7·60	+0·251601833850	−0·159213768396
7·21	+0·294513056583	−0·057197967799	7·61	+0·249998194750	−0·161510921566
7·22	+0·293926763993	−0·060058930954	7·62	+0·248371678346	−0·163789196464
7·23	+0·293311910786	−0·062910047738	7·63	+0·246722474402	−0·166048397306
7·24	+0·292668596729	−0·065751057450	7·64	+0·245050774627	−0·168288330341
7·25	+0·291996924192	−0·068581700653	7·65	+0·243356772660	−0·170508803876
7·26	+0·291296998131	−0·071401719205	7·66	+0·241640664046	−0·172709628281
7·27	+0·290568926079	−0·074210856276	7·67	+0·239902646217	−0·174890616014
7·28	+0·289812818129	−0·077008856374	7·68	+0·238142918467	−0·177051581630
7·29	+0·289028786922	−0·079795465364	7·69	+0·236361681936	−0·179192341800
7·30	+0·288216947635	−0·082570430493	7·70	+0·234559139586	−0·181312715325
7·31	+0·287377417963	−0·085333500412	7·71	+0·232735496182	−0·183412523148
7·32	+0·286510318111	−0·088084425194	7·72	+0·230890958266	−0·185491588374
7·33	+0·285615770772	−0·090822956363	7·73	+0·229025734139	−0·187549736279
7·34	+0·284693901119	−0·093548846906	7·74	+0·227140033840	−0·189586794329
7·35	+0·283744836788	−0·096261851305	7·75	+0·225234069120	−0·191602592189
7·36	+0·282768707860	−0·098961725549	7·76	+0·223308053424	−0·193596961740
7·37	+0·281765646852	−0·101648227162	7·77	+0·221362201866	−0·195569737092
7·38	+0·280735788696	−0·104321115218	7·78	+0·219396731209	−0·197520754596
7·39	+0·279679270724	−0·106980150367	7·79	+0·217411859839	−0·199449852859
7·40	+0·278596232657	−0·109625094854	7·80	+0·215407807746	−0·201356872756
7·41	+0·277486816584	−0·112255712538	7·81	+0·213384796501	−0·203241657440
7·42	+0·276351166945	−0·114871768912	7·82	+0·211343049230	−0·205104052360
7·43	+0·275189430519	−0·117473031128	7·83	+0·209282790594	−0·206943905267
7·44	+0·274001756407	−0·120059268011	7·84	+0·207204246765	−0·208761066232
7·45	+0·272788296009	−0·122630250080	7·85	+0·205107645402	−0·210555387651
7·46	+0·271549203014	−0·125185749572	7·86	+0·202993215628	−0·212326724262
7·47	+0·270284633379	−0·127725540456	7·87	+0·200861188009	−0·214014933156
7·48	+0·268994745315	−0·130249398456	7·88	+0·198711794526	−0·215799873784
7·49	+0·267679699262	−0·132757101068	7·89	+0·196545268555	−0·217501407969
7·50	+0·266339657880	−0·135248427580	7·90	+0·194361844841	−0·219179399922
7·51	+0·264974786027	−0·137723159089	7·91	+0·192161759476	−0·220833716244
7·52	+0·263585250739	−0·140181078522	7·92	+0·189945249872	−0·222464225941
7·53	+0·262171221215	−0·142621970654	7·93	+0·187712554741	−0·224070800436
7·54	+0·260732868795	−0·145045622124	7·94	+0·185463914068	−0·225653313572
7·55	+0·259270366946	−0·147451821455	7·95	+0·183199569087	−0·227211641627
7·56	+0·257783891239	−0·149840359071	7·96	+0·180919762257	−0·228745663321
7·57	+0·256273619329	−0·152211027316	7·97	+0·178624737238	−0·230255259825
7·58	+0·254739730943	−0·154563620468	7·98	+0·176314738866	−0·231740314769
7·59	+0·253182407850	−0·156897934760	7·99	+0·173990013128	−0·233200714254
7·60	+0·251601833850	−0·159213768396	8·00	+0·171650807138	−0·234636346854

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
8·00	+0·171650807138	−0·234636346854	8·40	+0·069157261657	−0·270786268277
8·01	+0·169297369111	−0·236047103631	8·41	+0·066447598160	−0·271141908453
8·02	+0·166929948339	−0·237432878137	8·42	+0·063734513946	−0·271470411269
8·03	+0·164548795169	−0·238793566425	8·43	+0·061018280395	−0·271771776141
8·04	+0·162154160970	−0·240129067056	8·44	+0·058299168877	−0·272046005084
8·05	+0·159746298117	−0·241439281101	8·45	+0·055577450731	−0·272293102707
8·06	+0·157325459958	−0·242724112158	8·46	+0·052853397237	−0·272513076214
8·07	+0·154891900797	−0·243983466348	8·47	+0·050127279588	−0·272705935396
8·08	+0·152445875859	−0·245217252327	8·48	+0·047399368869	−0·272871692631
8·09	+0·149987641274	−0·246425381291	8·49	+0·044669936026	−0·273010362878
8·10	+0·147517454044	−0·247607766982	8·50	+0·041939251843	−0·273121963674
8·11	+0·145035572024	−0·248764325692	8·51	+0·039207586917	−0·273206515132
8·12	+0·142542253891	−0·249894976273	8·52	+0·036475211629	−0·273264039934
8·13	+0·140037759122	−0·250999640134	8·53	+0·033742396123	−0·273294563325
8·14	+0·137522347965	−0·252078241253	8·54	+0·031009410275	−0·273298113112
8·15	+0·134996281417	−0·253130706180	8·55	+0·028276523672	−0·273274719657
8·16	+0·132459821198	−0·254156964039	8·56	+0·025544005583	−0·273224415870
8·17	+0·129913229721	−0·255156946534	8·57	+0·022812124938	−0·273147237207
8·18	+0·127356770071	−0·256130587952	8·58	+0·020081150296	−0·273043221660
8·19	+0·124790705977	−0·257077825169	8·59	+0·017351349826	−0·272912409756
8·20	+0·122215301784	−0·257998597649	8·60	+0·014622991279	−0·272754844546
8·21	+0·119630822433	−0·258892847451	8·61	+0·011896341961	−0·272570571599
8·22	+0·117037533429	−0·259760519231	8·62	+0·009171668713	−0·272359639000
8·23	+0·114435700818	−0·260601560243	8·63	+0·006449237878	−0·272122097337
8·24	+0·111825591161	−0·261415920344	8·64	+0·003729315286	−0·271857999697
8·25	+0·109207471506	−0·262203551993	8·65	+0·001012166219	−0·271567401658
8·26	+0·106581609366	−0·262964410256	8·66	−0·001701944606	−0·271250361281
8·27	+0·103948272687	−0·263698452805	8·67	−0·004412753067	−0·270906939104
8·28	+0·101307729828	−0·264405639923	8·68	−0·007119995658	−0·270537198130
8·29	+0·098660249531	−0·265085934502	8·69	−0·009823409518	−0·270141203821
8·30	+0·096006100895	−0·265739302042	8·70	−0·012522732450	−0·269719024092
8·31	+0·093345553353	−0·266365710658	8·71	−0·015217702949	−0·269270729296
8·32	+0·090678876643	−0·266965131077	8·72	−0·017908060228	−0·268796392222
8·33	+0·088006340781	−0·267537536636	8·73	−0·020593544236	−0·268296088078
8·34	+0·085328216040	−0·268082903285	8·74	−0·023273895691	−0·267769894490
8·35	+0·082644772917	−0·268601209586	8·75	−0·025948856095	−0·267217891486
8·36	+0·079956282113	−0·269092436712	8·76	−0·028618167764	−0·266640161489
8·37	+0·077263014501	−0·269556568447	8·77	−0·031281573850	−0·266036789304
8·38	+0·074565241107	−0·269993591184	8·78	−0·033938818366	−0·265407862113
8·39	+0·071863233078	−0·270403493925	8·79	−0·036589646207	−0·264753469460
8·40	+0·069157261657	−0·270786268277	8·80	−0·039233803177	−0·264073703240

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
8·80	−0·039233803177	−0·264073703240	9·20	−0·136748370765	−0·217408654960
8·81	−0·041871036007	−0·263368657691	9·21	−0·138914405500	−0·215795016778
8·82	−0·044501092388	−0·262638429381	9·22	−0·141064205893	−0·214161816342
8·83	−0·047123720982	−0·261883117196	9·23	−0·143197577219	−0·212509233706
8·84	−0·049738671456	−0·261102822332	9·24	−0·145314326565	−0·210837450612
8·85	−0·052345694498	−0·260297648278	9·25	−0·147414262841	−0·209146650470
8·86	−0·054944541843	−0·259467700807	9·26	−0·149497196801	−0·207437018341
8·87	−0·057534966296	−0·258613087962	9·27	−0·151562941057	−0·205708740917
8·88	−0·060116721752	−0·257733920049	9·28	−0·153611310096	−0·203962006501
8·89	−0·062689563221	−0·256830309615	9·29	−0·155642120296	−0·202197004987
8·90	−0·065253246851	−0·255902371444	9·30	−0·157655189943	−0·200413927844
8·91	−0·067807529947	−0·254950222539	9·31	−0·159650339244	−0·198612968091
8·92	−0·070352170997	−0·253973982110	9·32	−0·161627390345	−0·196794320281
8·93	−0·072886929689	−0·252973771561	9·33	−0·163586167343	−0·194958180481
8·94	−0·075411566939	−0·251949714476	9·34	−0·165526496306	−0·193104746248
8·95	−0·077925844909	−0·250901936605	9·35	−0·167448205283	−0·191234216615
8·96	−0·080429527028	−0·249830565850	9·36	−0·169351124322	−0·189346792063
8·97	−0·082922378016	−0·248735732253	9·37	−0·171235085481	−0·187442674507
8·98	−0·085404163904	−0·247617567976	9·38	−0·173099922846	−0·185522067274
8·99	−0·087874652054	−0·246476207294	9·39	−0·174945472543	−0·183585175079
9·00	−0·090333611183	−0·245311786573	9·40	−0·176771572752	−0·181632204007
9·01	−0·092780811380	−0·244124444261	9·41	−0·178578063718	−0·179663361493
9·02	−0·095216024131	−0·242914320868	9·42	−0·180364787772	−0·177678856298
9·03	−0·097639022336	−0·241681558953	9·43	−0·182131589336	−0·175678898489
9·04	−0·100049580330	−0·240426303111	9·44	−0·183878314938	−0·173663699419
9·05	−0·102447473906	−0·239148699952	9·45	−0·185604813228	−0·171633471704
9·06	−0·104832480333	−0·237848898088	9·46	−0·187310934989	−0·169588429202
9·07	−0·107204378374	−0·236527048119	9·47	−0·188996533147	−0·167528786993
9·08	−0·109562948310	−0·235183302612	9·48	−0·190661462784	−0·165454761353
9·09	−0·111907971956	−0·233817816088	9·49	−0·192305581154	−0·163366569738
9·10	−0·114239232683	−0·232430745006	9·50	−0·193928747687	−0·161264430758
9·11	−0·116556515436	−0·231022247743	9·51	−0·195530824010	−0·159148564154
9·12	−0·118859606752	−0·229592484581	9·52	−0·197111673948	−0·157019190783
9·13	−0·121148294781	−0·228141617686	9·53	−0·198671163543	−0·154876532586
9·14	−0·123422369306	−0·226669811094	9·54	−0·200209161060	−0·152720812575
9·15	−0·125681621757	−0·225177230692	9·55	−0·201725537001	−0·150552254803
9·16	−0·127925845233	−0·223664044201	9·56	−0·203220164114	−0·148371084348
9·17	−0·130154834519	−0·222130421159	9·57	−0·204692917400	−0·146177527286
9·18	−0·132368386105	−0·220576532901	9·58	−0·206143674127	−0·143971810670
9·19	−0·134566298203	−0·219002552542	9·59	−0·207572313841	−0·141754162508
9·20	−0·136748370765	−0·217408654960	9·60	−0·208978718369	−0·139524811741

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
9·60	−0·208978718369	−0·139524811741	10·00	−0·245935764451	−0·043472746169
9·61	−0·210362771833	−0·137283988215	10·01	−0·246357974862	−0·040969056455
9·62	−0·211724360660	−0·135031922668	10·02	−0·246755140400	−0·038463812722
9·63	−0·213063373585	−0·132768846695	10·03	−0·247127246760	−0·035957261846
9·64	−0·214379701667	−0·130494992737	10·04	−0·247474282103	−0·033449650599
9·65	−0·215673238291	−0·128210594048	10·05	−0·247796237059	−0·030941225625
9·66	−0·216943879179	−0·125915884679	10·06	−0·248093104724	−0·028432233416
9·67	−0·218191522398	−0·123611099451	10·07	−0·248364880658	−0·025922920290
9·68	−0·219416068367	−0·121296473933	10·08	−0·248611562881	−0·023413532364
9·69	−0·220617419863	−0·118972244417	10·09	−0·248833151876	−0·020904315537
9·70	−0·221795482032	−0·116638647900	10·10	−0·249029650581	−0·018395515458
9·71	−0·222950162390	−0·114295922054	10·11	−0·249201064392	−0·015887377509
9·72	−0·224081370836	−0·111944305207	10·12	−0·249347401155	−0·013380146780
9·73	−0·225189019654	−0·109584036317	10·13	−0·249468671167	−0·010874068044
9·74	−0·226273023521	−0·107215354950	10·14	−0·249564887171	−0·008369385737
9·75	−0·227333299512	−0·104838501258	10·15	−0·249636064351	−0·005866343931
9·76	−0·228369767107	−0·102453715952	10·16	−0·249682220330	−0·003365186314
9·77	−0·229382348196	−0·100061240280	10·17	−0·249703375168	−0·000866156165
9·78	−0·230370967084	−0·097661316004	10·18	−0·249699551355	+0·001630503669
9·79	−0·231335550495	−0·095254185376	10·19	−0·249670773804	+0·004124550795
9·80	−0·232276027579	−0·092840091113	10·20	−0·249617069854	+0·006615743298
9·81	−0·233192329916	−0·090419276375	10·21	−0·249538469258	+0·009103839761
9·82	−0·234084391517	−0·087991984743	10·22	−0·249435004182	+0·011588599292
9·83	−0·234952148834	−0·085558460188	10·23	−0·249306709197	+0·014069781546
9·84	−0·235795540759	−0·083118947058	10·24	−0·249153621275	+0·016547146743
9·85	−0·236614508629	−0·080673690044	10·25	−0·248975779783	+0·019020455697
9·86	−0·237408996230	−0·078222934162	10·26	−0·248773226477	+0·021489469834
9·87	−0·238178949800	−0·075766924729	10·27	−0·248546005495	+0·023953951217
9·88	−0·238924318032	−0·073305907338	10·28	−0·248294163353	+0·026413662567
9·89	−0·239645052073	−0·070840127831	10·29	−0·248017748933	+0·028868367285
9·90	−0·240341105535	−0·068369832284	10·30	−0·247716813482	+0·031317829476
9·91	−0·241012434487	−0·065895266972	10·31	−0·247391410602	+0·033761813968
9·92	−0·241658997463	−0·063416678354	10·32	−0·247041596243	+0·036200086339
9·93	−0·242280755465	−0·060934313045	10·33	−0·246667428695	+0·038632412933
9·94	−0·242877671958	−0·058448417794	10·34	−0·246268968580	+0·041058560885
9·95	−0·243449712877	−0·055959239457	10·35	−0·245846278846	+0·043478298146
9·96	−0·243996846626	−0·053467024979	10·36	−0·245399424757	+0·045891393496
9·97	−0·244519044079	−0·050972021363	10·37	−0·244928473884	+0·048297616575
9·98	−0·245016278580	−0·048474475654	10·38	−0·244433496098	+0·050696737897
9·99	−0·245488525942	−0·045974634906	10·39	−0·243914563561	+0·053088528877
10·00	−0·245935764451	−0·043472746169	10·40	−0·243371750714	+0·055472761849

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
10·40	−0·243371750714	+0·055472761849	10·80	−0·203201967112	+0·142166568299
10·41	−0·242805134273	+0·057849210087	10·81	−0·201770826005	+0·144058996415
10·42	−0·242214793214	+0·060217647828	10·82	−0·200320840603	+0·145935398812
10·43	−0·241600808767	+0·062577850293	10·83	−0·198852172014	+0·147795605727
10·44	−0·240963264405	+0·064929593703	10·84	−0·197364983034	+0·149639449122
10·45	−0·240302245833	+0·067272655308	10·85	−0·195859438131	+0·151466762702
10·46	−0·239617840978	+0·069606813400	10·86	−0·194335703428	+0·153277381926
10·47	−0·238910139979	+0·071931847339	10·87	−0·192793946683	+0·155071144022
10·48	−0·238179235177	+0·074247537568	10·88	−0·191234337275	+0·156847888004
10·49	−0·237425221101	+0·076553665638	10·89	−0·189657046181	+0·158607454682
10·50	−0·236648194462	+0·078850014227	10·90	−0·188062245963	+0·160349686681
10·51	−0·235848254136	+0·081136367158	10·91	−0·186450110748	+0·162074428448
10·52	−0·235025501155	+0·083412509421	10·92	−0·184820816208	+0·163781526274
10·53	−0·234180038696	+0·085678227191	10·93	−0·183174539542	+0·165470828298
10·54	−0·233311972068	+0·087933307849	10·94	−0·181511459461	+0·167142184528
10·55	−0·232421408701	+0·090177540002	10·95	−0·179831756165	+0·168795446850
10·56	−0·231508458131	+0·092410713500	10·96	−0·178135611325	+0·170430469041
10·57	−0·230573231989	+0·094632619458	10·97	−0·176423208066	+0·172047106783
10·58	−0·229615843992	+0·096843050272	10·98	−0·174694730946	+0·173645217675
10·59	−0·228636409922	+0·099041799642	10·99	−0·172950365937	+0·175224661243
10·60	−0·227635047621	+0·101228662586	11·00	−0·171190300407	+0·176785298957
10·61	−0·226611876971	+0·103403435462	11·01	−0·169414723099	+0·178326994235
10·62	−0·225567019886	+0·105565915987	11·02	−0·167623824113	+0·179849612465
10·63	−0·224500600296	+0·107715903254	11·03	−0·165817794883	+0·181353021005
10·64	−0·223412744130	+0·109853197747	11·04	−0·163996828161	+0·182837089204
10·65	−0·222303579310	+0·111977601366	11·05	−0·162161117996	+0·184301688406
10·66	−0·221173235728	+0·114088917441	11·06	−0·160310859712	+0·185746691967
10·67	−0·220021845238	+0·116186950748	11·07	−0·158446249891	+0·187171975260
10·68	−0·218849541635	+0·118271507531	11·08	−0·156567486350	+0·188577415689
10·69	−0·217656460650	+0·120342395515	11·09	−0·154674768122	+0·189962892696
10·70	−0·216442739924	+0·122399423927	11·10	−0·152768295436	+0·191328287775
10·71	−0·215208519001	+0·124442403513	11·11	−0·150848269694	+0·192673484480
10·72	−0·213953939309	+0·126471146550	11·12	−0·148914893455	+0·193998368432
10·73	−0·212679144146	+0·128485466871	11·13	−0·146968370410	+0·195302827334
10·74	−0·211384278663	+0·130485179874	11·14	−0·145008905360	+0·196586750976
10·75	−0·210069489850	+0·132470102543	11·15	−0·143036704202	+0·197850031243
10·76	−0·208734926518	+0·134440053463	11·16	−0·141051973900	+0·199092562127
10·77	−0·207380739286	+0·136394852837	11·17	−0·139054922470	+0·200314239736
10·78	−0·206007080560	+0·138334322500	11·18	−0·137045758956	+0·201514962299
10·79	−0·204614104523	+0·140258285937	11·19	−0·135024693407	+0·202694630176
10·80	−0·203201967112	+0·142166568299	11·20	−0·132991936860	+0·203853145865

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
11·20	−0·132991936860	+0·203853145865	11·60	−0·044615674094	+0·232000474620
11·21	−0·130947701315	+0·204990414012	11·61	−0·042294477301	+0·232235010376
11·22	−0·128892199715	+0·206106341416	11·62	−0·039971051364	+0·232446303109
11·23	−0·126825645926	+0·207200837037	11·63	−0·037645628720	+0·232634351719
11·24	−0·124748254710	+0·208273812006	11·64	−0·035318441806	+0·232799157379
11·25	−0·122660241711	+0·209325179625	11·65	−0·032989723038	+0·232940723529
11·26	−0·120561823424	+0·210354855380	11·66	−0·030659704782	+0·233059055883
11·27	−0·118453217184	+0·211362756947	11·67	−0·028328619340	+0·233154162418
11·28	−0·116334641133	+0·212348804193	11·68	−0·025996698919	+0·233226053376
11·29	−0·114206314208	+0·213312919188	11·69	−0·023664175616	+0·233274741260
11·30	−0·112068456110	+0·214255026208	11·70	−0·021331281388	+0·233300240831
11·31	−0·109921287289	+0·215175051739	11·71	−0·018998248037	+0·233302569105
11·32	−0·107765028918	+0·216072924488	11·72	−0·016665307180	+0·233281745349
11·33	−0·105599902872	+0·216948575381	11·73	−0·014332690232	+0·233237791079
11·34	−0·103426131706	+0·217801937572	11·74	−0·012000628381	+0·233170730054
11·35	−0·101243938632	+0·218632946448	11·75	−0·009669352567	+0·233080588274
11·36	−0·099053547496	+0·219441539632	11·76	−0·007339093458	+0·232967393973
11·37	−0·096855182759	+0·220227656988	11·77	−0·005010081428	+0·232831177619
11·38	−0·094649069469	+0·220991240623	11·78	−0·002682546537	+0·232671971904
11·39	−0·092435433245	+0·221732234896	11·79	−0·000356718505	+0·232489811743
11·40	−0·090214500248	+0·222450586415	11·80	+0·001967173307	+0·232284734267
11·41	−0·087986497163	+0·223146244045	11·81	+0·004288899920	+0·232056778820
11·42	−0·085751651176	+0·223819158911	11·82	+0·006608232761	+0·231805986948
11·43	−0·083510189950	+0·224469284397	11·83	+0·008924943683	+0·231532402401
11·44	−0·081262341601	+0·225096576153	11·84	+0·011238804987	+0·231236071121
11·45	−0·079008334679	+0·225700992096	11·85	+0·013549589443	+0·230917041237
11·46	−0·076748398145	+0·226282492413	11·86	+0·015857070317	+0·230575363062
11·47	−0·074482761342	+0·226841039560	11·87	+0·018161021385	+0·230211089083
11·48	−0·072211653982	+0·227376598268	11·88	+0·020461216961	+0·229824273953
11·49	−0·069935306115	+0·227889135543	11·89	+0·022757431916	+0·229414974489
11·50	−0·067653948112	+0·228378620665	11·90	+0·025049441700	+0·228983249662
11·51	−0·065367810637	+0·228845025194	11·91	+0·027337022362	+0·228529160587
11·52	−0·063077124631	+0·229288322968	11·92	+0·029619950574	+0·228052770520
11·53	−0·060782121280	+0·229708490101	11·93	+0·031898003653	+0·227554144849
11·54	−0·058483032003	+0·230105504990	11·94	+0·034170959578	+0·227033351083
11·55	−0·056180088419	+0·230479348310	11·95	+0·036438597013	+0·226490458847
11·56	−0·053873522332	+0·230830003018	11·96	+0·038700695332	+0·225925539874
11·57	−0·051563565704	+0·231157454348	11·97	+0·040957034634	+0·225338667993
11·58	−0·049250450632	+0·231461689817	11·98	+0·043207395768	+0·224729919124
11·59	−0·046934409328	+0·231742699216	11·99	+0·045451560353	+0·224099371266
11·60	−0·044615674094	+0·232000474620	12·00	+0·047689310797	+0·223447104491

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
12·00	+0·047689310797	+0·223447104491	12·40	+0·129561026518	+0·180710246883
12·01	+0·049920430320	+0·222773200930	12·41	+0·131360894344	+0·179260532985
12·02	+0·052144702973	+0·222077744768	12·42	+0·133146181728	+0·177794184461
12·03	+0·054361913660	+0·221360822234	12·43	+0·134916723111	+0·176311359192
12·04	+0·056571848157	+0·220622521586	12·44	+0·136672354521	+0·174812216550
12·05	+0·058774293132	+0·219862933107	12·45	+0·138412913587	+0·173296917383
12·06	+0·060969036167	+0·219082149091	12·46	+0·140138239554	+0·171765624000
12·07	+0·063155865777	+0·218280263834	12·47	+0·141848173298	+0·170218500152
12·08	+0·065334571427	+0·217457373624	12·48	+0·143542557339	+0·168655711017
12·09	+0·067504943560	+0·216613576726	12·49	+0·145221235856	+0·167077423179
12·10	+0·069666773607	+0·215748973377	12·50	+0·146884054700	+0·165483804615
12·11	+0·071819854013	+0·214863665770	12·51	+0·148530861410	+0·163875024675
12·12	+0·073963978255	+0·213957758045	12·52	+0·150161505225	+0·162251254066
12·13	+0·076098940860	+0·213031356277	12·53	+0·151775837096	+0·160612664833
12·14	+0·078224537427	+0·212084568463	12·54	+0·153373709704	+0·158959430343
12·15	+0·080340564642	+0·211117504511	12·55	+0·154954977468	+0·157291725265
12·16	+0·082446820302	+0·210130276228	12·56	+0·156519496560	+0·155609725554
12·17	+0·084543103331	+0·209122997309	12·57	+0·158067124921	+0·153913608430
12·18	+0·086629213798	+0·208095783320	12·58	+0·159597722266	+0·152203552365
12·19	+0·088704952938	+0·207048751691	12·59	+0·161111150104	+0·150479737058
12·20	+0·090770123171	+0·205982021700	12·60	+0·162607271746	+0·148742343422
12·21	+0·092824528115	+0·204895714458	12·61	+0·164085952318	+0·146991553564
12·22	+0·094867972612	+0·203789952902	12·62	+0·165547058774	+0·145227550765
12·23	+0·096900262741	+0·202664861776	12·63	+0·166990459905	+0·143450519461
12·24	+0·098921205837	+0·201520567620	12·64	+0·168416026353	+0·141660645228
12·25	+0·100930610511	+0·200357198756	12·65	+0·169823630622	+0·139858114759
12·26	+0·102928286663	+0·199174885273	12.66	+0·171213147086	+0·138043115846
12·27	+0·104914045507	+0·197973759015	12·67	+0·172584452006	+0·136215837361
12·28	+0·106887699579	+0·196753953565	12·68	+0·173937423535	+0·134376469238
12·29	+0·108849062765	+0·195515604234	12·69	+0·175271941729	+0·132525202454
12·30	+0·110797950308	+0·194258848041	12·70	+0·176587888562	+0·130662229004
12·31	+0·112734178832	+0·192983823702	12·71	+0·177885147930	+0·128787741891
12·32	+0·114657566356	+0·191690671617	12·72	+0·179163605667	+0·126901935099
12·33	+0·116567932311	+0·190379533851	12·73	+0·180423149549	+0·125005003575
12·34	+0·118465097559	+0·189050554121	12·74	+0·181663669309	+0·123097143211
12·35	+0·120348884405	+0·187703877780	12·75	+0·182885056640	+0·121178550823
12·36	+0·122219116616	+0·186339651802	12·76	+0·184087205211	+0·119249424132
12·37	+0·124075619437	+0·184958024768	12·77	+0·185270010670	+0·117309961743
12·38	+0·125918219608	+0·183559146848	12·78	+0·186433370658	+0·115360363124
12·39	+0·127746745377	+0·182143169785	12·79	+0·187577184813	+0·113400828590
12·40	+0·129561026518	+0·180710246883	12·80	+0·188701354781	+0·111431559278

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
12·80	+0·188701354781	+0·111431559278	13·20	+0·216685922259	+0·027066702765
12·81	+0·189805784222	+0·109452757129	13·21	+0·216945650832	+0·024878857605
12·82	+0·190890378823	+0·107464624869	13·22	+0·217183496687	+0·022690195350
12·83	+0·191955046298	+0·105467365986	13·23	+0·217399452738	+0·020500932874
12·84	+0·192999696401	+0·103461184712	13·24	+0·217593514066	+0·018311286951
12·85	+0·194024240934	+0·101446286001	13·25	+0·217765677921	+0·016121474234
12·86	+0·195028593748	+0·099422875508	13·26	+0·217915943717	+0·013931711237
12·87	+0·196012670759	+0·097391159571	13·27	+0·218044313033	+0·011742214308
12·88	+0·196976389945	+0·095351345187	13·28	+0·218150789610	+0·009553199615
12·89	+0·197919671360	+0·093303639994	13·29	+0·218235379352	+0·007364883118
12·90	+0·198842437136	+0·091248252250	13·30	+0·218298090319	+0·005177480555
12·91	+0·199744611493	+0·089185390809	13·31	+0·218338932728	+0·002991207414
12·92	+0·200626120738	+0·087115265106	13·32	+0·218357918950	+0·000806278917
12·93	+0·201486893280	+0·085038085131	13·33	+0·218355063505	−0·001377090000
12·94	+0·202326859628	+0·082954061409	13·34	+0·218330383064	−0·003558684713
12·95	+0·203145952399	+0·080863404982	13·35	+0·218283896439	−0·005738290927
12·96	+0·203944106324	+0·078766327385	13·36	+0·218215624587	−0·007915694697
12·97	+0·204721258250	+0·076663040627	13·37	+0·218125590599	−0·010090682449
12·98	+0·205477347147	+0·074553757168	13·38	+0·218013819702	−0·012263041002
12·99	+0·206212314114	+0·072438689899	13·39	+0·217880339252	−0·014432557586
13·00	+0·206926102377	+0·070318052122	13·40	+0·217725178732	−0·016599019864
13·01	+0·207618657300	+0·068192057526	13·41	+0·217548369742	−0·018762215954
13·02	+0·208289926385	+0·066060920168	13·42	+0·217349946004	−0·020921934445
13·03	+0·208939859276	+0·063924854454	13·43	+0·217129943348	−0·023077964423
13·04	+0·209568407762	+0·061784075111	13·44	+0·216888399712	−0·025230095486
13·05	+0·210175525783	+0·059638797173	13·45	+0·216625355135	−0·027378117768
13·06	+0·210761169428	+0·057489235957	13·46	+0·216340851750	−0·029521821957
13·07	+0·211325296943	+0·055335607039	13·47	+0·216034933785	−0·031660999316
13·08	+0·211867868729	+0·053178126239	13·48	+0·215707647547	−0·033795441703
13·09	+0·212388847348	+0·051017009592	13·49	+0·215359041426	−0·035924941590
13·10	+0·212888197522	+0·048852473334	13·50	+0·214989165880	−0·038049292086
13·11	+0·213365886137	+0·046684733877	13·51	+0·214598073436	−0·040168286951
13·12	+0·213821882244	+0·044514007788	13·52	+0·214185818679	−0·042281720622
13·13	+0·214256157060	+0·042340511767	13·53	+0·213752458244	−0·044389388228
13·14	+0·214668683969	+0·040164462629	13·54	+0·213298050815	−0·046491085613
13·15	+0·215059438525	+0·037986077278	13·55	+0·212822657111	−0·048586609352
13·16	+0·215428398451	+0·035805572692	13·56	+0·212326339882	−0·050675756773
13·17	+0·215775543638	+0·033623165893	13·57	+0·211809163903	−0·052758325976
13·18	+0·216100856151	+0·031439073935	13·58	+0·211271195961	−0·054834115851
13·19	+0·216404320223	+0·029253513878	13·59	+0·210712504851	−0·056902926099
13·20	+0·216685922259	+0·027066702765	13·60	+0·210133161369	−0·058964557249

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
13·60	+0·210133161369	−0·058964557249	14·00	+0·171073476110	−0·133375154699
13·61	+0·209533238299	−0·061018810678	14·01	+0·169731671331	−0·134983384921
13·62	+0·208912810407	−0·063065488629	14·02	+0·168373856986	−0·136577042971
13·63	+0·208271954434	−0·065104394233	14·03	+0·167000179537	−0·138155981458
13·64	+0·207610749084	−0·067135331522	14·04	+0·165610786908	−0·139720054543
13·65	+0·206929275015	−0·069158105453	14·05	+0·164205828478	−0·141269117950
13·66	+0·206227614833	−0·071172521923	14·06	+0·162785455058	−0·142803028980
13·67	+0·205505853079	−0·073178387788	14·07	+0·161349818877	−0·144321646527
13·68	+0·204764076220	−0·075175510884	14·08	+0·159899073571	−0·145824831084
13·69	+0·204002372641	−0·077163700040	14·09	+0·158433374159	−0·147312444762
13·70	+0·203220832633	−0·079142765100	14·10	+0·156952877033	−0·148784351297
13·71	+0·202419548383	−0·081112516941	14·11	+0·155457739939	−0·150240416070
13·72	+0·201598613965	−0·083072767489	14·12	+0·153948121961	−0·151680506109
13·73	+0·200758125328	−0·085023329736	14·13	+0·152424183503	−0·153104490110
13·74	+0·199898180285	−0·086964017760	14·14	+0·150886086277	−0·154512238442
13·75	+0·199018878503	−0·088894646742	14·15	+0·149333993280	−0·155903623164
13·76	+0·198120321493	−0·090815032981	14·16	+0·147768068780	−0·157278518033
13·77	+0·197202612595	−0·092724993914	14·17	+0·146188478301	−0·158636798515
13·78	+0·196265856970	−0·094624348132	14·18	+0·144595388601	−0·159978341800
13·79	+0·195310161589	−0·096512915397	14·19	+0·142988967659	−0·161303026807
13·80	+0·194335635216	−0·098390516658	14·20	+0·141369384657	−0·162610734200
13·81	+0·193342388402	−0·100256974070	14·21	+0·139736809960	−0·163901346396
13·82	+0·192330533469	−0·102112111008	14·22	+0·138091415099	−0·165174747575
13·83	+0·191300184501	−0·103955752084	14·23	+0·136433372759	−0·166430823692
13·84	+0·190251457328	−0·105787723166	14·24	+0·134762856750	−0·167669462485
13·85	+0·189184469514	−0·107607851391	14·25	+0·133080042002	−0·168890553486
13·86	+0·188099340348	−0·109415965181	14·26	+0·131385104536	−0·170093988031
13·87	+0·186996190826	−0·111211894262	14·27	+0·129678221452	−0·171279659270
13·88	+0·185875143642	−0·112995469678	14·28	+0·127959570912	−0·172447462171
13·89	+0·184736323171	−0·114766523805	14·29	+0·126229332114	−0·173597293538
13·90	+0·183579855458	−0·116524890369	14·30	+0·124487685284	−0·174729052013
13·91	+0·182405868205	−0·118270404461	14·31	+0·122734811649	−0·175842638087
13·92	+0·181214490755	−0·120002902550	14·32	+0·120970893423	−0·176937954108
13·93	+0·180005854081	−0·121722222501	14·33	+0·119196113786	−0·178014904291
13·94	+0·178780090769	−0·123428203590	14·34	+0·117410656869	−0·179073394724
13·95	+0·177537335004	−0·125120686515	14·35	+0·115614707731	−0·180113333378
13·96	+0·176277722558	−0·126799513414	14·36	+0·113808452342	−0·181134630112
13·97	+0·175001390777	−0·128464527879	14·37	+0·111992077563	−0·182137196684
13·98	+0·173708478559	−0·130115574971	14·38	+0·110165771130	−0·183120946756
13·99	+0·172399126347	−0·131752501232	14·39	+0·108329721631	−0·184085795902
14·00	+0·171073476110	−0·133375154699	14·40	+0·106484118490	−0·185031661615

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
14·40	+0·106484118490	−0·185031661615	14·80	+0·027082314586	−0·206595567180
14·41	+0·104629151946	−0·185958463314	14·81	+0·025015737179	−0·206716471994
14·42	+0·102765013033	−0·186866122350	14·82	+0·022948053986	−0·206816724913
14·43	+0·100891893564	−0·187754562014	14·83	+0·020879471508	−0·206896329814
14·44	+0·099009986107	−0·188623707542	14·84	+0·018810196197	−0·206955292607
14·45	+0·097119483970	−0·189473486119	14·85	+0·016740434436	−0·206993621235
14·46	+0·095220581177	−0·190303826889	14·86	+0·014670392520	−0·207011325670
14·47	+0·093313472454	−0·191114660960	14·87	+0·012600276630	−0·207008417910
14·48	+0·091398353204	−0·191905921406	14·88	+0·010530292822	−0·206984911980
14·49	+0·089475419488	−0·192677543276	14·89	+0·008460646998	−0·206940823925
14·50	+0·087544868010	−0·193429463596	14·90	+0·006391544891	−0·206876171810
14·51	+0·085606896092	−0·194161621377	14·91	+0·004323192042	−0·206790975716
14·52	+0·083661701655	−0·194873957618	14·92	+0·002255793783	−0·206685257736
14·53	+0·081709483202	−0·195566415311	14·93	+0·000189555214	−0·206559041974
14·54	+0·079750439794	−0·196238939443	14·94	−0·001875318817	−0·206412354539
14·55	+0·077784771035	−0·196891477005	14·95	−0·003938623732	−0·206245223541
14·56	+0·075812677046	−0·197523976991	14·96	−0·006000155243	−0·206057679091
14·57	+0·073834358450	−0·198136390405	14·97	−0·008059709376	−0·205849753289
14·58	+0·071850016350	−0·198728670261	14·98	−0·010117082484	−0·205621480228
14·59	+0·069859852307	−0·199300771592	14·99	−0·012172071276	−0·205372895984
14·60	+0·067864068323	−0·199852651447	15·00	−0·014224472827	−0·205104038614
14·61	+0·065862866820	−0·200384268898	15·01	−0·016274084604	−0·204814948148
14·62	+0·063856450617	−0·200895585039	15·02	−0·018320704486	−0·204505666588
14·63	+0·061845022913	−0·201386562994	15·03	−0·020364130779	−0·204176237900
14·64	+0·059828787267	−0·201857167913	15·04	−0·022404162240	−0·203826708006
14·65	+0·057807947575	−0·202307366980	15·05	−0·024440598094	−0·203457124785
14·66	+0·055782708050	−0·202737129411	15·06	−0·026473238057	−0·203067538060
14·67	+0·053753273205	−0·203146426455	15·07	−0·028501882349	−0·202657999596
14·68	+0·051719847828	−0·203535231400	15·08	−0·030526331722	−0·202228563094
14·69	+0·049682636966	−0·203903519571	15·09	−0·032546387470	−0·201779284182
14·70	+0·047641845902	−0·204251268330	15·10	−0·034561851456	−0·201310220408
14·71	+0·045597680133	−0·204578457081	15·11	−0·036572526126	−0·200821431239
14·72	+0·043550345355	−0·204885067267	15·12	−0·038578214533	−0·200312978045
14·73	+0·041500047438	−0·205171082373	15·13	−0·040578720351	−0·199784924098
14·74	+0·039446992407	−0·205436487924	15·14	−0·042573847897	−0·199237334565
14·75	+0·037391386420	−0·205681271486	15·15	−0·044563402147	−0·198670276496
14·76	+0·035333435752	−0·205905422669	15·16	−0·046547188761	−0·198083818818
14·77	+0·033273346769	−0·206108933120	15·17	−0·048525014094	−0·197478032331
14·78	+0·031211325913	−0·206291796530	15·18	−0·050496685220	−0·196852989694
14·79	+0·029147579677	−0·206454008627	15·19	−0·052462009949	−0·196208765420
14·80	+0·027082314586	−0·206595567180	15·20	−0·054420796844	−0·195545435866

x	$J_0(x)$	$-J_1(x)$	x	$J_0(x)$	$-J_1(x)$
15·20	−0·054420796844	−0·195545435866	15·35	−0·082890403582	−0·183360322017
15·21	−0·056372855242	−0·194863079227	15·36	−0·084719235661	−0·182403162448
15·22	−0·058317995271	−0·194161775523	15·37	−0·086538408385	−0·181428468883
15·23	−0·060256027869	−0·193441606594	15·38	−0·088347746952	−0·180436349242
15·24	−0·062186764798	−0·192702656088	15·39	−0·090147077648	−0·179426913096
15·25	−0·064110018670	−0·191945009455	15·40	−0·091936227862	−0·178400271655
15·26	−0·066025602957	−0·191168753932	15·41	−0·093715026106	−0·177356537757
15·27	−0·067933332015	−0·190373978539	15·42	−0·095483302024	−0·176295825856
15·28	−0·069833021097	−0·189560774066	15·43	−0·097240886416	−0·175218252010
15·29	−0·071724486374	−0·188729233063	15·44	−0·098987611250	−0·174123933866
15·30	−0·073607544951	−0·187879449832	15·45	−0·100723309676	−0·173012990652
15·31	−0·075482014884	−0·187011520415	15·46	−0·102447816048	−0·171885543160
15·32	−0·077347715198	−0·186125542581	15·47	−0·104160965933	−0·170741713736
15·33	−0·079204465905	−0·185221615823	15·48	−0·105862596129	−0·169581626266
15·34	−0·081052088022	−0·184299841336	15·49	−0·107552544683	−0·168405406163
15·35	−0·082890403582	−0·183360322017	15·50	−0·109230650900	−0·167213180352

TABLE II.

n	$J_n(1)$
0	+0·76519 76865 57966 551
1	+0·44005 05857 44933 516
2	+0·11490 34849 31900 480
3	+0·01956 33539 82668 406
4	+0·00247 66389 64109 955
5	+0·00024 97577 30211 234
6	+0·00002 09383 38002 389
7	15023 25817 437
8	00942 23441 726
9	00052 49250 180
10	+0· 00002 63061 512
11	11980 067
12	00499 972
13	00019 256
14	689
15	023
16	001

n	$J_n(2)$
0	+0·22389 07791 41235 668
1	+0·57672 48077 56873 387
2	+0·35283 40286 15637 719
3	+0·12894 32494 74402 051
4	+0·03399 57198 07568 434
5	+0·00703 96297 55871 685
6	+0·00120 24289 71789 993
7	+0·00017 49440 74868 274
8	+0·00002 21795 52287 926
9	24923 43435 133
10	+0· 02515 38628 272
11	00230 42847 584
12	00019 32695 149
13	00001 49494 201
14	10729 463
15	00718 302
16	00045 060
17	00002 659
18	148
19	008

n	$J_n(3)$
0	−0·26005 19549 01933 438
1	+0·33905 89585 25936 459
2	+0·48609 12605 85891 077
3	+0·30906 27222 55251 644
4	+0·13203 41839 24612 210
5	+0·04302 84348 77047 584
6	+0·01139 39323 32213 069
7	+0·00254 72944 51804 694
8	+0·00049 34417 76208 835
9	+0·00008 43950 21309 092
10	+0·00001 29283 51645 716
11	17939 89662 347
12	02275 72544 832
13	00265 90696 309
14	00028 80156 513
15	00002 90764 476
16	27488 250
17	02443 521
18	00204 983
19	00016 280
20	+0· 00001 228
21	088
22	006

n	$J_n(4)$
0	−0·39714 98098 63847 372
1	−0·06604 33280 23549 136
2	+0·36412 81458 52072 804
3	+0·43017 14738 75621 940
4	+0·28112 90649 61360 106
5	+0·13208 66560 47098 272
6	+0·04908 75751 56385 574
7	+0·01517 60694 22058 451
8	+0·00402 86678 20819 004
9	+0·00093 86018 61217 564
10	+0·00019 50405 54660 035
11	+0·00003 66009 12082 608
12	62644 61794 312
13	09858 58683 265
14	01436 19646 909
15	00194 78845 096
16	00024 71691 311
17	00002 94685 392
18	33134 523
19	03525 313
20	+0· 00355 951
21	00034 199
22	00003 134
23	275
24	023
25	002

n	$J_n(5)$
0	−0·17759 67713 14338 304
1	−0·32757 91375 91465 222
2	+0·04656 51162 77752 216
3	+0·36483 12306 13666 994
4	+0·39123 26304 58648 178
5	+0·26114 05461 20170 090
6	+0·13104 87317 81692 002
7	+0·05337 64101 55890 715
8	+0·01840 52166 54802 001
9	+0·00552 02831 39475 688
10	+0·00146 78026 47310 474
11	+0·00035 09274 49766 209
12	+0·00007 62781 31660 846
13	+0·00001 52075 82205 849
14	28012 95809 572
15	04796 74327 752
16	00767 50156 939
17	00115 26676 659
18	00016 31244 339
19	00002 18282 584
20	+0· 27703 301
21	03343 820
22	00384 787
23	00042 309
24	00004 454
25	450
26	044
27	004

n	$J_n(6)$
0	+0·15064 52572 50996 932
1	−0·27668 38581 27565 608
2	−0·24287 32099 60185 468
3	+0·11476 83848 20775 296
4	+0·35764 15947 80960 764
5	+0·36208 70748 87172 389
6	+0·24583 68633 64326 551
7	+0·12958 66518 41480 713
8	+0·05653 19909 32461 779
9	+0·02116 53239 78417 365
10	+0·00696 39810 02790 316
11	+0·00204 79460 30883 689
12	+0·00054 51544 43783 211
13	+0·00013 26717 44249 154
14	+0·00002 97564 47963 121
15	61916 79578 746
16	12019 49930 610
17	02187 20051 176
18	00374 63692 719
19	00060 62105 141
20	+0· 00009 29639 841
21	00001 35493 798
22	18816 747
23	02496 677
24	00316 779
25	00038 554
26	00004 415
27	507
28	055
29	006
30	+0· 001

n	$J_n(7)$
0	+0·30007 92705 19555 597
1	−0·00468 28234 82345 833
2	−0·30141 72200 85940 120
3	−0·16755 55879 95334 236
4	+0·15779 81446 61367 918
5	+0·34789 63247 51183 285
6	+0·33919 66049 83179 632
7	+0·23358 35695 05696 084
8	+0·12797 05340 28212 537
9	+0·05892 05082 73075 428
10	+0·02353 93443 88267 135
11	+0·00833 47614 07687 815
12	+0·00265 56200 35894 568
13	+0·00077 02215 72522 133
14	+0·00020 52029 47759 069
15	+0·00005 05902 18514 143
16	+0·00001 16122 74444 403
17	24944 64660 269
18	05036 96762 619
19	00959 75833 201
20	+0· 00173 14903 330
21	00029 66471 543
22	00004 83925 930
23	75348 588
24	11221 932
25	01601 804
26	00219 522
27	00028 933
28	00003 673
29	450
30	+0· 053
31	006
32	001

n	$J_n(8)$
0	+0·17165 08071 37553 906
1	+0·23463 63468 53914 624
2	−0·11299 17204 24075 250
3	−0·29113 22070 65952 249
4	−0·10535 74348 75388 937
5	+0·18577 47721 90563 312
6	+0·33757 59001 13593 077
7	+0·32058 90779 79826 304
8	+0·22345 49863 51102 954
9	+0·12632 08947 22379 605
10	+0·06076 70267 74251 156
11	+0·02559 66722 13248 286
12	+0·00962 38218 12181 630
13	+0·00327 47932 23296 605
14	+0·00101 92561 63532 336
15	+0·00029 26033 49066 572
16	+0·00007 80063 95467 308
17	+0·00001 94222 32802 661
18	45380 93944 002
19	09991 89945 347
20	+0· 02080 58296 397
21	00411 01536 639
22	00077 24770 956
23	00013 84703 619
24	00002 37274 853
25	38945 500
26	06134 520
27	00928 879
28	00135 416
29	00019 034
30	+0· 00002 583
31	339
32	043
33	005
34	001

n	$J_n(9)$
0	−0·09033 36111 82876 134
1	+0·24531 17865 73325 272
2	+0·14484 73415 32503 973
3	−0·18093 51903 36656 840
4	−0·26547 08017 56941 866
5	−0·05503 88556 69513 708
6	+0·20431 65176 79704 413
7	+0·32746 08792 42452 925
8	+0·30506 70722 53000 137
9	+0·21488 05825 40658 430
10	+0·12469 40928 28316 722
11	+0·06221 74015 22267 619
12	+0·02739 28886 70559 681
13	+0·01083 03015 99224 863
14	+0·00389 46492 82756 591
15	+0·00128 63850 58240 087
16	+0·00039 33009 11377 031
17	+0·00011 20181 82211 578
18	+0·00002 98788 88088 932
19	74973 70144 148
20	+0· 17766 74741 915
21	03989 62042 141
22	00851 48121 408
23	00173 17662 520
24	00033 64375 918
25	00006 25675 712
26	00001 11600 257
27	19125 771
28	03154 368
29	00501 407
30	+0· 00076 922
31	00011 403
32	00001 636
33	227
34	031
35	004

n	$J_n(10)$
0	−0·24593 57644 51348 335
1	+0·04347 27461 68861 437
2	+0·25463 03136 85120 623
3	+0·05837 93793 05186 812
4	−0·21960 26861 02008 535
5	−0·23406 15281 86793 640
6	−0·01445 88420 84785 105
7	+0·21671 09176 85051 514
8	+0·31785 41268 43857 225
9	+0·29185 56852 65120 046
10	+0·20748 61066 33358 858
11	+0·12311 65280 01597 669
12	+0·06337 02549 70156 015
13	+0·02897 20839 26776 767
14	+0·01195 71632 39463 579
15	+0·00450 79731 43721 253
16	+0·00156 67561 91700 181
17	+0·00050 56466 69719 325
18	+0·00015 24424 85345 524
19	+0·00004 31462 77524 563
20	+0·00001 15133 69247 813
21	29071 99466 691
22	06968 68512 289
23	01590 21987 380
24	00346 32629 661
25	00072 14634 990
26	00014 40545 292
27	00002 76200 527
28	50937 552
29	09049 767
30	+0· 01551 096
31	00256 809
32	00041 123
33	00006 376
34	958
35	140
36	020
37	003

n	$J_n(11)$
0	−0·17119 03004 07196 088
1	−0·17678 52989 56721 501
2	+0·13904 75187 78701 270
3	+0·22734 80330 58067 417
4	−0·01503 95007 47028 133
5	−0·23828 58517 83178 787
6	−0·20158 40008 74043 491
7	+0·01837 60326 47858 615
8	+0·22497 16787 89499 910
9	+0·30885 55001 36868 527
10	+0·28042 82305 25375 862
11	+0·20101 40099 09269 403
12	+0·12159 97892 93162 945
13	+0·06429 46212 75813 386
14	+0·03036 93155 40577 785
15	+0·01300 90910 09293 703
16	+0·00511 00235 75677 768
17	+0·00185 64321 19950 713
18	+0·00062 80393 40533 526
19	+0·00019 89693 58159 009
20	+0·00005 93093 51288 506
21	+0·00001 67010 10162 830
22	44581 42060 481
23	11315 58079 093
24	02738 28088 453
25	00633 28125 065
26	00140 27025 479
27	00029 81449 927
28	00006 09183 254
29	00001 19846 638
30	+0· 22735 384
31	04164 546
32	00737 509
33	00126 418
34	00020 997
35	00003 383
36	529
37	080
38	012
39	002

n	$J_n(12)$
0	+0·04768 93107 96833 537
1	−0·22344 71044 90627 612
2	−0·08493 04948 78604 805
3	+0·19513 69395 31092 677
4	+0·18249 89646 44151 144
5	−0·07347 09631 01658 581
6	−0·24372 47672 28866 628
7	−0·17025 38041 27208 047
8	+0·04509 53290 80457 240
9	+0·23038 09095 67817 701
10	+0·30047 60352 71269 311
11	+0·27041 24825 50964 484
12	+0·19528 01827 38832 243
13	+0·12014 78829 26700 003
14	+0·06504 02302 69017 762
15	+0·03161 26543 67674 776
16	+0·01399 14056 50169 178
17	+0·00569 77606 99443 032
18	+0·00215 22496 64919 412
19	+0·00075 89882 95315 204
20	+0·00025 12132 70245 400
21	+0·00007 83892 72169 462
22	+0·00002 31491 82347 716
23	64910 63105 497
24	17332 26223 355
25	04418 41787 923
26	01077 81226 324
27	00252 10192 815
28	00056 64641 343
29	00012 24800 120
30	+0· 00002 55225 904
31	51329 401
32	09976 003
33	01875 946
34	00341 699
35	00060 351
36	00010 346
37	00001 723
38	279
39	044
40	+0· 007
41	001

n	$J_n(13)$
0	+0·20692 61023 77067 811
1	−0·07031 80521 21778 371
2	−0·21774 42642 41956 791
3	+0·00331 98169 70407 051
4	+0·21927 64874 59067 738
5	+0·13161 95599 27480 788
6	−0·11803 06721 30236 362
7	−0·24057 09495 86160 507
8	−0·14104 57351 16398 030
9	+0·06697 61986 73670 624
10	+0·23378 20102 03018 894
11	+0·29268 84324 07896 905
12	+0·26153 68754 10345 099
13	+0·19014 88760 41970 970
14	+0·11876 08766 73596 841
15	+0·06564 37814 08852 996
16	+0·03272 47727 31448 533
17	+0·01490 95053 14712 625
18	+0·00626 93180 91646 024
19	+0·00245 16832 46768 672
20	+0·00089 71406 29677 786
21	+0·00030 87494 59932 207
22	+0·00010 03576 25487 806
23	+0·00003 09225 03257 290
24	90604 62961 066
25	25315 13829 722
26	06761 28691 713
27	01730 00937 128
28	00424 90585 590
29	00100 35431 567
30	+0· 00022 82878 324
31	00005 00929 928
32	00001 06172 104
33	21763 505
34	04319 539
35	00831 008
36	00155 121
37	00028 122
38	00004 956
39	850
40	+0· 142
41	023
42	004
43	001

n	$J_n(14)$
0	+0·17107 34761 10458 659
1	+0·13337 51546 98793 253
2	−0·15201 98825 82059 623
3	−0·17680 94068 65096 003
4	+0·07624 44224 97018 479
5	+0·22037 76482 91963 705
6	+0·08116 81834 25812 739
7	−0·15080 49196 41267 072
8	−0·23197 31030 67079 810
9	−0·11430 71981 49681 283
10	+0·08500 67054 46061 018
11	+0·23574 53487 86911 308
12	+0·28545 02712 19085 324
13	+0·25359 79733 02949 247
14	+0·18551 73934 86391 849
15	+0·11743 68136 69834 451
16	+0·06613 29215 20396 260
17	+0·03372 41498 05357 001
18	+0·01576 85851 49756 457
19	+0·00682 36405 79731 031
20	+0·00275 27249 95227 770
21	+0·00104 12879 78062 597
22	+0·00037 11389 38960 020
23	+0·00012 51486 87240 324
24	+0·00004 00638 90543 902
25	+0·00001 22132 23195 912
26	35547 63727 213
27	09901 84933 738
28	02645 21017 203
29	00678 99135 075
30	+0· 00167 75399 538
31	00039 95434 356
32	00009 18666 897
33	00002 04185 745
34	43923 044
35	09154 753
36	01850 722
37	00363 244
38	00069 281
39	00012 851
40	+0· 00002 320
41	408
42	070
43	012
44	002

n	$J_n(15)$
0	−0·01422 44728 26780 773
1	+0·20510 40386 13522 761
2	+0·04157 16779 75250 475
3	−0·19401 82578 20122 635
4	−0·11917 89811 03299 529
5	+0·13045 61345 65029 553
6	+0·20614 97374 79985 897
7	+0·03446 36554 18959 165
8	−0·17398 36590 88957 343
9	−0·22004 62251 13846 998
10	−0·09007 18110 47659 054
11	+0·09995 04770 50301 592
12	+0·23666 58440 54768 056
13	+0·27871 48734 37327 297
14	+0·24643 99365 69932 593
15	+0·18130 63414 93213 542
16	+0·11617 27464 16494 492
17	+0·06652 88508 61974 707
18	+0·03462 59822 03981 511
19	+0·01657 35064 27580 920
20	+0·00736 02340 79223 485
21	+0·00305 37844 50348 374
22	+0·00119 03623 81751 963
23	+0·00043 79452 02790 717
24	+0·00015 26695 73472 902
25	+0·00005 05974 32322 570
26	+0·00001 59885 34268 998
27	48294 86476 623
28	13976 17046 846
29	03882 83831 601
30	+0· 01037 47102 011
31	00267 04576 442
32	00066 31813 951
33	00015 91163 081
34	00003 69303 606
35	83013 267
36	18091 639
37	03826 599
38	00786 251
39	00157 074
40	+0· 00030 535
41	00005 781
42	00001 067
43	192
44	034
45	006
46	001

n	$J_n(16)$
0	−0·17489 90739 83629 185
1	+0·09039 71756 61304 186
2	+0·18619 87209 41292 208
3	−0·04384 74954 25981 134
4	−0·20264 15317 26035 133
5	−0·05747 32704 37036 433
6	+0·16672 07377 02887 363
7	+0·18251 38237 14201 955
8	−0·00702 11419 52960 653
9	−0·18953 49656 67162 607
10	−0·20620 56944 22597 281
11	−0·06822 21523 61083 994
12	+0·11240 02349 26106 790
13	+0·23682 25047 50244 178
14	+0·27243 63352 93040 000
15	+0·23994 10820 12575 821
16	+0·17745 31934 80539 665
17	+0·11496 53049 48503 509
18	+0·06684 80795 35030 292
19	+0·03544 28740 05314 648
20	+0·01732 87462 27591 996
21	+0·00787 89915 63665 343
22	+0·00335 36066 27029 529
23	+0·00134 34266 60665 861
24	+0·00050 87450 22384 822
25	+0·00018 28084 06488 605
26	+0·00006 25312 47892 069
27	+0·00002 04181 49160 619
28	63800 05525 020
29	19118 70176 952
30	+0· 05505 23866 431
31	01525 49322 163
32	00407 79131 952
33	00105 22205 645
34	00026 24966 335
35	00006 33901 280
36	00001 48351 763
37	33681 654
38	07425 886
39	01591 305
40	+0· 00331 726
41	00067 325
42	00013 313
43	00002 567
44	483
45	089
46	016
47	003

n	$J_n(17)$
0	−0·16985 42521 51183 548
1	−0·09766 84927 57780 650
2	+0·15836 38412 38503 471
3	+0·13493 05730 49193 232
4	−0·11074 12860 44670 566
5	−0·18704 41194 23155 851
6	+0·00071 53334 42814 183
7	+0·18754 90606 76907 039
8	+0·15373 68341 73462 202
9	−0·04285 55696 90119 084
10	−0·19911 33197 27705 938
11	−0·19139 53946 95417 314
12	−0·04857 48381 13422 350
13	+0·12281 91526 52938 702
14	+0·23641 58951 12034 482
15	+0·26657 17334 13941 622
16	+0·23400 48109 12568 380
17	+0·17390 79106 56775 329
18	+0·11381 10104 00982 277
19	+0·06710 36407 80598 906
20	+0·03618 53631 08591 747
21	+0·01803 83900 63146 381
22	+0·00838 00711 65064 018
23	+0·00365 12058 93489 902
24	+0·00149 96624 29085 127
25	+0·00058 31350 82750 457
26	+0·00021 54407 55475 041
27	+0·00007 58601 69290 845
28	+0·00002 55268 41095 878
29	82282 48436 754
30	+0· 25460 06511 871
31	07576 56899 262
32	02172 12767 790
33	00600 85285 358
34	00160 59516 541
35	00041 52780 805
36	00010 40169 125
37	00002 52641 372
38	59563 905
39	13644 321
40	+0· 03039 452
41	00658 981
42	00139 163
43	+0·00000 00000 00028 646
44	00005 752
45	00001 127
46	216
47	040
48	007
49	001

n	$J_n(18)$
0	−0·01335 58057 21984 111
1	−0·18799 48854 88069 594
2	−0·00753 25148 87801 400
3	+0·18632 09932 90780 394
4	+0·06963 95126 51394 864
5	−0·15537 00987 79049 343
6	−0·15595 62341 95311 166
7	+0·05139 92759 82175 233
8	+0·19593 34488 48114 125
9	+0·12276 37896 60592 878
10	−0·07316 96591 87521 246
11	−0·20406 34109 80060 930
12	−0·17624 11764 54775 446
13	−0·03092 48242 92972 998
14	+0·13157 19858 09370 005
15	+0·23559 23577 74215 227
16	+0·26108 19438 14322 041
17	+0·22855 33201 17912 845
18	+0·17062 98830 75068 889
19	+0·11270 64460 32224 933
20	+0·06730 59474 37405 969
21	+0·03686 23260 50899 443
22	+0·01870 61466 81359 398
23	+0·00886 38102 81312 419
24	+0·00394 58129 26439 006
25	+0·00165 83575 22524 930
26	+0·00066 07357 47241 354
27	+0·00025 04346 36172 316
28	+0·00009 05681 61275 594
29	+0·00003 13329 76685 088

n	$J_n(18)$
30	+0·00001 03936 52487 466
31	33125 31606 465
32	10161 78601 469
33	03005 47865 425
34	00858 30238 423
35	00236 99701 950
36	00063 35269 160
37	00016 41374 689
38	00004 12604 562
39	00001 00733 463
40	+0· 23907 111
41	05520 363
42	01241 210
43	00271 949
44	00058 104
45	00012 114
46	00002 466
47	490
48	095
49	018
50	+0· 003

n	$J_n(19)$
0	+0·14662 94396 59651 204
1	−0·10570 14311 42409 267
2	−0·15775 59060 95694 285
3	+0·07248 96614 38052 575
4	+0·18064 73781 28763 519
5	+0·00357 23925 10900 486
6	−0·17876 71715 44079 053
7	−0·11647 79745 38739 888
8	+0·09294 12955 68165 452
9	+0·19474 43287 01405 531

n	$J_n(19)$
10	+0·09155 33316 22639 788
11	−0·09837 24006 77574 175
12	−0·20545 82166 17725 675
13	−0·16115 37676 81658 257
14	−0·01506 79917 88754 044
15	+0·13894 83060 98231 244
16	+0·23446 00540 49119 166
17	+0·25593 17849 31864 194
18	+0·22352 31400 39479 918
19	+0·16758 57435 63992 493
20	+0·11164 83470 88505 067
21	+0·06746 34082 01281 333
22	+0·03748 12920 93274 722
23	+0·01933 53734 88407 496
24	+0·00933 06647 73396 058
25	+0·00423 68322 54908 861
26	+0·00181 88937 92153 576
27	+0·00074 11928 60458 822
28	+0·00028 76543 37571 496
29	+0·00010 66304 50278 219
30	+0·00003 78491 42225 174
31	+0·00001 28931 56748 644
32	42232 64007 245
33	13325 74644 181
34	04056 79493 594
35	01193 30911 839
36	00339 60707 917
37	00093 62297 110
38	00025 02975 563
39	00006 49605 144
40	+0· 00001 63824 500
41	40182 226
42	09593 529
43	02231 272
44	00505 913
45	00111 905
46	00024 163
47	00005 096
48	00001 051
49	212
50	+0· 042
51	008
52	002

n	$J_n(20)$
0	+0·16702 46643 40583 155
1	+0·06683 31241 75850 046
2	−0·16034 13519 22998 150
3	−0·09890 13945 60449 676
4	+0·13067 09335 54863 247
5	+0·15116 97679 82394 975
6	−0·05508 60495 63665 760
7	−0·18422 13977 20594 431
8	−0·07386 89288 40750 341
9	+0·12512 62546 47994 158
10	+0·18648 25580 23945 083
11	+0·06135 63033 75950 926
12	−0·11899 06243 10399 065
13	−0·20414 50525 48429 804
14	−0·14639 79440 02559 680
15	−0·00081 20690 55153 748
16	+0·14517 98404 19829 058
17	+0·23309 98137 26880 240
18	+0·25108 98429 15867 351
19	+0·21886 19035 21680 991
20	+0·16474 77737 75326 532
21	+0·11063 36440 28972 073
22	+0·06758 28786 85514 822
23	+0·03804 86890 79160 535
24	+0·01992 91061 96554 408
25	+0·00978 11657 92570 045
26	+0·00452 38082 84870 704
27	+0·00198 07357 48093 786
28	+0·00082 41782 34982 517
29	+0·00032 69633 09857 262
30	+0·00012 40153 63603 543
31	+0·00004 50827 80953 368
32	+0·00001 57412 57351 896
33	52892 42572 701
34	17132 43138 017
35	05357 84096 556
36	01620 01199 928
37	00474 20223 186
38	00134 53625 859
39	00037 03555 077
40	+0· 00009 90238 941
41	00002 57400 689
42	65103 882
43	16035 615
44	03849 264
45	00901 145
46	+0·00000 00000 00205 887
47	00045 937
48	00010 015
49	00002 135
50	+0· 445
51	091
52	018
53	004
54	001

n	$J_n(21)$
0	+0·03657 90710 00862 743
1	+0·17112 02727 63900 104
2	−0·02028 19021 66205 590
3	−0·17498 34922 24129 740
4	−0·02971 33813 26402 907
5	+0·16366 41088 61690 537
6	+0·10764 86712 60541 258
7	−0·10215 05824 27095 533
8	−0·17574 90595 45271 613
9	−0·03175 34629 40730 458
10	+0·14853 18055 96074 078
11	+0·17321 23254 13181 961
12	+0·03292 87257 89164 167
13	−0·13557 94959 39851 484
14	−0·20078 90540 95646 957
15	−0·13213 92428 54344 458
16	+0·01201 87071 60869 159
17	+0·15045 34632 89954 606
18	+0·23157 26143 56200 203
19	+0·24652 81613 20674 313
20	+0·21452 59632 71686 649
21	+0·16209 27211 01585 971
22	+0·10965 94789 31485 294
23	+0·06766 99966 59621 310
24	+0·03857 00375 61018 529
25	+0·02049 00891 94135 328
26	+0·01021 58890 91684 632
27	+0·00480 63980 80512 332
28	+0·00214 34202 58204 222
29	+0·00090 93892 74698 928

n	$J_n(21)$
30	+0·00036 82263 10011 863
31	+0·00014 26858 96763 539
32	+0·00005 30368 13766 205
33	+0·00001 89501 07095 370
34	65206 65676 388
35	21644 29380 552
36	06940 98925 453
37	02153 38363 859
38	00647 12451 956
39	00188 59081 313
40	+0· 00053 35564 351
41	00014 66878 120
42	00003 92245 451
43	00001 02103 682
44	25893 439
45	06402 159
46	01544 385
47	00363 716
48	00083 679
49	00018 818
50	+0· 00004 139
51	891
52	188
53	039
54	008
55	002

n	$J_n(22)$
0	−0·12065 14757 04867 180
1	+0·11717 77896 43851 701
2	+0·13130 40020 36126 426
3	−0·09330 43347 28192 351
4	−0·15675 06387 80178 885
5	+0·03630 41024 44490 938
6	+0·17325 25035 27674 766
7	+0·05819 72631 16058 934
8	−0·13621 78815 44728 171
9	−0·15726 48133 30406 695
10	+0·00754 66706 38031 784
11	+0·16412 54230 01344 681

n	$J_n(22)$
12	+0·15657 87523 63312 897
13	+0·00668 77613 94996 661
14	−0·14867 50343 51044 115
15	−0·19591 05323 87234 626
16	−0·11847 56916 31548 557
17	+0·02358 22536 50436 726
18	+0·15492 09927 27678 042
19	+0·22992 48253 58490 979
20	+0·24222 18874 36988 195
21	+0·21047 86063 45123 920
22	+0·15960 09064 94612 017
23	+0·10872 32066 44100 114
24	+0·06772 94346 70324 584
25	+0·03905 01053 63880 797
26	+0·02102 08047 93040 864
27	+0·01063 54332 37852 155
28	+0·00508 43495 18050 788
29	+0·00230 65473 53549 852
30	+0·00099 65480 50398 821
31	+0·00041 13109 65719 660
32	+0·00016 26010 34811 129
33	+0·00006 17102 26458 170
34	+0·00002 25296 44563 381
35	79268 56737 734
36	26921 72329 410
37	08838 89067 607
38	02809 09079 813
39	00865 24117 203
40	+0· 00258 58244 815
41	00075 05863 943
42	00021 18157 153
43	00005 81645 187
44	00001 55546 760
45	40541 854
46	10306 278
47	02557 128
48	00619 634
49	00146 728
50	+0· 00033 973
51	00007 696
52	00001 706
53	370
54	079
55	016
56	003
57	001

n	$J_n(23)$	n	$J_n(23)$
0	−0·16241 27813 13486 542	30	+0·00246 97721 07261 064
1	−0·03951 93218 83701 511	31	+0·00108 54000 46200 881
2	+0·15897 63185 40990 759	32	+0·00045 60888 86845 660
3	+0·06716 73772 82134 687	33	+0·00018 37168 56326 172
4	−0·14145 43940 32607 797	34	+0·00007 10986 13916 400
5	−0·11636 89056 41302 616	35	+0·00002 64877 41339 705
6	+0·09085 92176 66824 051	36	95162 51030 530
7	+0·16377 37148 58776 034	37	33022 61886 301
8	+0·00882 91305 08083 100	38	11084 17647 134
9	−0·15763 17110 27066 051	39	03603 35556 401
10	−0·13219 30782 68395 662	40	+0· 01135 89891 967
11	+0·04268 12081 84982 867	41	00347 59720 006
12	+0·17301 85817 49683 622	42	00103 36066 315
13	+0·13785 99205 97295 695	43	00029 89391 752
14	−0·01717 69323 78827 619	44	00008 41659 366
15	−0·15877 09687 10651 057	45	00002 30870 169
16	−0·18991 56355 04630 281	46	61745 644
17	−0·10545 94806 87095 422	47	16112 408
18	+0·03401 90118 80228 354	48	04105 067
19	+0·15870 66297 17018 062	49	01021 783
20	+0·22819 19415 65279 749	50	+0· 00248 619
21	+0·23814 89208 31294 545	51	00059 168
22	+0·20668 86964 74475 507	52	00013 780
23	+0·15725 55419 89441 207	53	00003 142
24	+0·10782 23875 04406 908	54	702
25	+0·06776 50928 02364 513	55	154
26	+0·03949 30316 31168 121	56	034
27	+0·02152 35004 50711 239	57	007
28	+0·01104 04042 09632 179	58	001
29	+0·00535 74837 11871 458		

n	$J_n(24)$
0	−0·05623 02741 66859 267
1	−0·15403 80651 83121 221
2	+0·04339 37687 34932 499
3	+0·16127 03599 72276 638
4	−0·00307 61787 41863 339
5	−0·16229 57528 86231 084
6	−0·06454 70516 27399 613
7	+0·13002 22270 72531 278
8	+0·14039 33507 53042 858
9	−0·03642 66599 03836 039
10	−0·16771 33456 80919 887
11	−0·10333 44614 96930 534
12	+0·07299 00893 08733 565
13	+0·17632 45508 05664 098
14	+0·11802 81740 64069 208
15	−0·03862 50143 97583 355
16	−0·16630 94420 61048 403
17	−0·18312 09083 50481 181
18	−0·09311 18447 68799 938
19	+0·04345 31411 97281 275
20	+0·16191 26516 64495 289
21	+0·22640 12782 43544 208
22	+0·23428 95852 61707 074
23	+0·20312 96280 69585 428
24	+0·15504 22018 71664 996
25	+0·10695 47756 73744 565
26	+0·06778 02474 48636 180
27	+0·03990 24271 31633 826
28	+0·02200 02135 97539 927
29	+0·01143 14045 95959 338

n	$J_n(24)$
30	+0·00562 56808 42695 140
31	+0·00263 27975 10778 513
32	+0·00117 57127 26816 017
33	+0·00050 24364 27397 533
34	+0·00020 59874 48527 199
35	+0·00008 11946 76762 864
36	+0·00003 08303 58697 822
37	+0·00001 12963 99330 601
38	40002 05904 866
39	13709 19368 140
40	+0· 04552 82041 591
41	01466 87437 162
42	00459 00035 379
43	00139 62686 664
44	00041 32925 168
45	00011 91372 284
46	00003 34720 896
47	91724 484
48	24533 335
49	06408 854
50	+0· 01636 153
51	00408 451
52	00099 762
53	00023 852
54	00005 585
55	00001 281
56	288
57	064
58	014
59	003
60	+0· 001

TABLE III.

The first fifty roots of $J_1(x)=0$, with the corresponding maximum or minimum values of $J_0(x)$.

No. of root (n)	Value of root (x_n)	$J_0(x_n)=\frac{\text{Min.}}{\text{Max.}}$	No. of root (n)	Value of root (x_n)	$J_0(x_n)=\frac{\text{Max.}}{\text{Min.}}$
1	3·8317 0597 0207 5123	−0·4027 5939 5702 5547	26	82·4622 5991 4373 5565	+0·0878 6187 6039 4105
2	7·0155 8666 9815 6188	+0·3001 1575 2526 1326	27	85·6040 1943 6350 2310	−0·0862 3466 3413 2884
3	10·1734 6813 5062 7221	−0·2497 0487 7057 8259	28	88·7457 6714 4926 3069	+0·0846 9463 4803 7192
4	13·3236 9193 6314 2231	+0·2183 5940 7247 8730	29	91·8875 0425 1694 9853	−0·0832 3427 2981 9746
5	16·4706 3005 0877 6328	−0·1964 6537 1468 6572	30	95·0292 3180 8044 6953	+0·0818 4693 7926 4857
6	19·6158 5851 0468 2420	+0·1800 6337 5344 3156	31	98·1709 5073 0790 7820	−0·0805 2673 9448 4029
7	22·7600 8438 0592 7719	−0·1671 8460 0473 8180	32	101·3126 6182 3038 7301	+0·0792 6843 1724 5187
8	25·9036 7208 7618 3826	+0·1567 2498 6252 8622	33	104·4543 6579 1282 7601	−0·0780 6732 5407 9485
9	29·0468 2853 4016 8551	−0·1480 1110 9972 7775	34	107·5960 6325 9509 1722	+0·0769 1921 3961 3909
10	32·1896 7991 0974 4036	+0·1406 0579 8193 1148	35	110·7377 5478 0899 2151	−0·0758 2031 1569 1671
11	35·3323 0755 0083 8651	−0·1342 1124 0310 0007	36	113·8794 4084 7594 9981	+0·0747 6720 0537 0746
12	38·4747 6623 4771 6151	+0·1286 1662 2072 0700	37	117·0211 2189 8892 4250	−0·0737 5678 6512 8573
13	41·6170 9421 2814 4509	−0·1236 6796 0769 8371	38	120·1627 9832 8149 0038	+0·0727 8626 0189 2388
14	44·7593 1899 7652 8217	+0·1192 4981 2010 6895	39	123·3044 7048 8635 7180	−0·0718 5306 4408 8473
15	47·9014 6088 7185 4471	−0·1152 7369 4120 1680	40	126·4461 3869 8516 5957	+0·0709 5486 5793 0974
16	51·0435 3518 3571 5095	+0·1116 7049 6859 2113	41	129·5878 0324 5103 9968	−0·0700 8953 0177 2614
17	54·1855 5364 1061 3205	−0·1083 8534 8943 6825	42	132·7294 6438 8509 6159	+0·0692 5510 1263 7661
18	57·3275 2543 7901 0107	+0·1053 7405 5395 2352	43	135·8711 2236 4789 0006	−0·0684 4978 2005 1879
19	60·4694 5784 5347 4916	−0·1026 0056 7103 3972	44	139·0127 7738 8659 7042	+0·0676 7191 8315 5457
20	63·6113 5669 8481 2326	+0·1000 3514 6811 5233	45	142·1544 2965 5859 0290	−0·0669 1998 4772 3973
21	66·7532 2673 4098 4934	−0·0976 5301 5783 1733	46	145·2960 7934 5195 9072	+0·0661 9257 2028 7533
22	69·8950 7183 7495 7740	+0·0954 3333 9020 5353	47	148·4377 2662 0342 2304	−0·0654 8837 5698 2572
23	73·0368 9522 5573 8348	−0·0933 5845 3290 4550	48	151·5793 7163 1401 4280	+0·0648 0618 6514 0981
24	76·1786 9958 4641 4576	+0·0914 1327 2155 9213	49	154·7210 1451 6285 9535	−0·0641 4488 1592 6670
25	79·3204 8717 5476 2994	−0·0895 8482 1964 8557	50	157·8626 5540 1930 2978	+0·0635 0341 6658 3216

x	$J_0(x\sqrt{i}) = X - Yi$	
	X	Y
0·0	+1·000000000	Nil
0·2	+0·999975000	+0·009999972
0·4	+0·999600004	+0·039998222
0·6	+0·997975114	+0·089979750
0·8	+0·993601138	+0·159886230
1·0	+0·984381781	+0·249566040
1·2	+0·967629156	+0·358704420
1·4	+0·940075057	+0·486733934
1·6	+0·897891139	+0·632725677
1·8	+0·836721794	+0·795261955
2·0	+0·751734183	+0·972291627
2·2	+0·637690457	+1·160969944
2·4	+0·489047772	+1·357485476
2·6	+0·300092090	+1·556877774
2·8	+0·065112108	+1·752850564
3·0	−0·221380250	+1·937586785
3·2	−0·564376430	+2·101573388
3·4	−0·968038995	+2·233445750
3·6	−1·435305322	+2·319863655
3·8	−1·967423273	+2·345433061
4·0	−2·563416557	+2·292690323
4·2	−3·219479832	+2·142167987
4·4	−3·928306622	+1·872563796
4·6	−4·678356937	+1·461036836
4·8	−5·453076175	+0·883656854
5·0	−6·230082479	+0·116034382
5·2	−6·980346403	−0·865839727
5·4	−7·667394351	−2·084516693
5·6	−8·246575962	−3·559746593
5·8	−8·664445263	−5·306844640
6·0	−8·858315966	−7·334746541

x	$I_1(x)$	x	$I_1(x)$	x	$I_1(x)$	x	$I_1(x)$
·00	Nil	·45	0·230743570	·90	0·497126448	1·35	0·840904230
·01	0·005000063	·46	0·236137373	·91	0·503751599	1·36	0·849809949
·02	0·010000500	·47	0·241548938	·92	0·510414946	1·37	0·858780872
·03	0·015001687	·48	0·246978674	·93	0·517117001	1·38	0·867817710
·04	0·020004000	·49	0·252426993	·94	0·523858282	1·39	0·876921172
·05	0·025007814	·50	0·257894304	·95	0·530639310	1·40	0·886091981
·06	0·030013502	·51	0·263381026	·96	0·537460608	1·41	0·895330860
·07	0·035021441	·52	0·268887571	·97	0·544322705	1·42	0·904638540
·08	0·040032009	·53	0·274414358	·98	0·551226129	1·43	0·914015758
·09	0·045045577	·54	0·279961803	·99	0·558171417	1·44	0·923463255
·10	0·050062526	·55	0·285530329	1·00	0·565159104	1·45	0·932981780
·11	0·055083230	·56	0·291120360	1·01	0·572189733	1·46	0·942572087
·12	0·060108065	·57	0·296732318	1·02	0·579263847	1·47	0·952234935
·13	0·065137410	·58	0·302366629	1·03	0·586381997	1·48	0·961971092
·14	0·070171639	·59	0·308023722	1·04	0·593544734	1·49	0·971781330
·15	0·075211135	·60	0·313704026	1·05	0·600752614	1·50	0·981666428
·16	0·080256272	·61	0·319407973	1·06	0·608006196	1·51	0·991627170
·17	0·085307432	·62	0·325135997	1·07	0·615306043	1·52	1·001664351
·18	0·090364993	·63	0·330888532	1·08	0·622652724	1·53	1·011778765
·19	0·095429332	·64	0·336666018	1·09	0·630046810	1·54	1·021971216
·20	0·100500834	·65	0·342468895	1·10	0·637488876	1·55	1·032242518
·21	0·105579878	·66	0·348297605	1·11	0·644979503	1·56	1·042593488
·22	0·110666843	·67	0·354152590	1·12	0·652519270	1·57	1·053024951
·23	0·115762116	·68	0·360034297	1·13	0·660108769	1·58	1·063537735
·24	0·120866075	·69	0·365943176	1·14	0·667748588	1·59	1·074132681
·25	0·125979109	·70	0·371879677	1·15	0·675439326	1·60	1·084810635
·26	0·131101599	·71	0·377844255	1·16	0·683181582	1·61	1·095572447
·27	0·136233930	·72	0·383837364	1·17	0·690975960	1·62	1·106418977
·28	0·141376489	·73	0·389859461	1·18	0·698823068	1·63	1·117351091
·29	0·146529663	·74	0·395911007	1·19	0·706723524	1·64	1·128369664
·30	0·151693840	·75	0·401992463	1·20	0·714677942	1·65	1·139475574
·31	0·156869409	·76	0·408104296	1·21	0·722686944	1·66	1·150669712
·32	0·162056756	·77	0·414246975	1·22	0·730751160	1·67	1·161952973
·33	0·167256278	·78	0·420420971	1·23	0·738871219	1·68	1·173326261
·34	0·172468361	·79	0·426626755	1·24	0·747047758	1·69	1·184790486
·35	0·177693400	·80	0·432864802	1·25	0·755281420	1·70	1·196346565
·36	0·182931789	·81	0·439135593	1·26	0·763572846	1·71	1·207995429
·37	0·188183922	·82	0·445439607	1·27	0·771922691	1·72	1·219738009
·38	0·193450196	·83	0·451777329	1·28	0·780331610	1·73	1·231575249
·39	0·198731008	·84	0·458149245	1·29	0·788800263	1·74	1·243508096
·40	0·204026756	·85	0·464555845	1·30	0·797329314	1·75	1·255537513
·41	0·209337840	·86	0·470997619	1·31	0·805919438	1·76	1·267664463
·42	0·214664660	·87	0·477475069	1·32	0·814571307	1·77	1·279889923
·43	0·220007618	·88	0·483988688	1·33	0·823285603	1·78	1·292214874
·44	0·225367121	·89	0·490538979	1·34	0·832063015	1·79	1·304640310

x	$I_1(x)$	x	$I_1(x)$	x	$I_1(x)$	x	$I_1(x)$
1·80	1·317167230	2·25	2·003967457	2·70	3·016107694	3·15	4·525620649
1·81	1·329796644	2·26	2·022411151	2·71	3·043474850	3·16	4·566596009
1·82	1·342529568	2·27	2·041014722	2·72	3·071086362	3·17	4·607943508
1·83	1·355367027	2·28	2·059779695	2·73	3·098944528	3·18	4·649666635
1·84	1·368310061	2·29	2·078707611	2·74	3·127051673	3·19	4·691768912
1·85	1·381359709	2·30	2·097800028	2·75	3·155410139	3·20	4·734253895
1·86	1·394517026	2·31	2·117058510	2·76	3·184022290	3·21	4·777125171
1·87	1·407783076	2·32	2·136484642	2·77	3·212890513	3·22	4·820386363
1·88	1·421158927	2·33	2·156080021	2·78	3·242017219	3·23	4·864041126
1·89	1·434645663	2·34	2·175846257	2·79	3·271404837	3·24	4·908093153
1·90	1·448244373	2·35	2·195784977	2·80	3·301055823	3·25	4·952546165
1·91	1·461956157	2·36	2·215897825	2·81	3·330972651	3·26	4·997403925
1·92	1·475782125	2·37	2·236186453	2·82	3·361157821	3·27	5·042670227
1·93	1·489723395	2·38	2·256652534	2·83	3·391613857	3·28	5·088348897
1·94	1·503781096	2·39	2·277297753	2·84	3·422343306	3·29	5·134443807
1·95	1·517956370	2·40	2·298123813	2·85	3·453348735	3·30	5·180588856
1·96	1·532250362	2·41	2·319132429	2·86	3·484632737	3·31	5·227897983
1·97	1·546664233	2·42	2·340325336	2·87	3·516197933	3·32	5·275265168
1·98	1·561199148	2·43	2·361704281	2·88	3·548046962	3·33	5·323064420
1·99	1·575856293	2·44	2·383271029	2·89	3·580182492	3·34	5·371299790
2·00	1·590636855	2·45	2·405027363	2·90	3·612607212	3·35	5·419975369
2·01	1·605542033	2·46	2·426975075	2·91	3·645323840	3·36	5·469095281
2·02	1·620573039	2·47	2·449115981	2·92	3·678335120	3·37	5·518663697
2·03	1·635731095	2·48	2·471451912	2·93	3·711643814	3·38	5·568684817
2·04	1·651017434	2·49	2·493984712	2·94	3·745252718	3·39	5·619162888
2·05	1·666433299	2·50	2·516716246	2·95	3·779164648	3·40	5·670102192
2·06	1·681979944	2·51	2·539648394	2·96	3·813382452	3·41	5·721507056
2·07	1·697658635	2·52	2·562783055	2·97	3·847908999	3·42	5·773381845
2·08	1·713470648	2·53	2·586122143	2·98	3·882747188	3·43	5·825730963
2·09	1·729417273	2·54	2·609667592	2·99	3·917899943	3·44	5·878558859
2·10	1·745499810	2·55	2·633421351	3·00	3·953370217	3·45	5·931870019
2·11	1·761719567	2·56	2·657385389	3·01	3·989160991	3·46	5·985668980
2·12	1·778077871	2·57	2·681561694	3·02	4·025275271	3·47	6·039960312
2·13	1·794576055	2·58	2·705952269	3·03	4·061716094	3·48	6·094748632
2·14	1·811215465	2·59	2·730559137	3·04	4·098486520	3·49	6·150038601
2·15	1·827997461	2·60	2·755384341	3·05	4·135589648	3·50	6·205834922
2·16	1·844923415	2·61	2·780429941	3·06	4·173028594	3·51	6·262142346
2·17	1·861994709	2·62	2·805698017	3·07	4·210806510	3·52	6·318965664
2·18	1·879212738	2·63	2·831190666	3·08	4·248926577	3·53	6·376309712
2·19	1·896578912	2·64	2·856910009	3·09	4·287392003	3·54	6·434179377
2·20	1·914094651	2·65	2·882858180	3·10	4·326206027	3·55	6·492579585
2·21	1·931761388	2·66	2·909037340	3·11	4·365371921	3·56	6·551515315
2·22	1·949580572	2·67	2·935449665	3·12	4·404892984	3·57	6·610991589
2·23	1·967553660	2·68	2·962097349	3·13	4·444772545	3·58	6·671013473
2·24	1·985682127	2·69	2·988982613	3·14	4·485013970	3·59	6·731586089

x	$I_1(x)$	x	$I_1(x)$	x	$I_1(x)$	x	$I_1(x)$
3·60	6·792714601	4·00	9·759465154	4·40	14·046221338	4·80	20·252834600
3·61	6·854404223	4·01	9·848494681	4·41	14·174997247	4·81	20·439443796
3·62	6·916660219	4·02	9·938347267	4·42	14·304970189	4·82	20·627795525
3·63	6·979487901	4·03	10·029030650	4·43	14·436151440	4·83	20·817906249
3·64	7·042892632	4·04	10·120552634	4·44	14·568552384	4·84	21·009792573
3·65	7·106879825	4·05	10·212921103	4·45	14·702184510	4·85	21·203471276
3·66	7·171454946	4·06	10·306144016	4·46	14·837059420	4·86	21·398859282
3·67	7·236623510	4·07	10·400229397	4·47	14·973188822	4·87	21·596273684
3·68	7·302391084	4·08	10·495185359	4·48	15·110584538	4·88	21·795431735
3·69	7·368763288	4·09	10·591020085	4·49	15·249258499	4·89	21·996450853
3·70	7·435745797	4·10	10·687741837	4·50	15·389222754	4·90	22·199348620
3·71	7·503344337	4·11	10·785358956	4·51	15·530489464	4·91	22·404142793
3·72	7·571564687	4·12	10·883879856	4·52	15·673070904	4·92	22·610851286
3·73	7·640412684	4·13	10·983313038	4·53	15·816979464	4·93	22·819492189
3·74	7·709894216	4·14	11·083667081	4·54	15·962227657	4·94	23·030083764
3·75	7·780015230	4·15	11·184950646	4·55	16·108828111	4·95	23·242644448
3·76	7·850781728	4·16	11·287172471	4·56	16·256793575	4·96	23·457192854
3·77	7·922199767	4·17	11·390341384	4·57	16·406136918	4·97	23·673747769
3·78	7·994275465	4·18	11·494466292	4·58	16·556871133	4·98	23·892328160
3·79	8·067014991	4·19	11·599556184	4·59	16·709009334	4·99	24·112953174
3·80	8·140424579	4·20	11·705620143	4·60	16·862564762	5·00	24·335642142
3·81	8·214510518	4·21	11·812667328	4·61	17·017550780	5·01	24·560414578
3·82	8·289279159	4·22	11·920706992	4·62	17·173980885	5·02	24·787290180
3·83	8·364736907	4·23	12·029748470	4·63	17·331868690	5·03	25·016288837
3·84	8·440890236	4·24	12·139801191	4·64	17·491227953	5·04	25·247430624
3·85	8·517745677	4·25	12·250874666	4·65	17·652072549	5·05	25·480735808
3·86	8·595309818	4·26	12·362978507	4·66	17·814416491	5·06	25·716224854
3·87	8·673589318	4·27	12·476122406	4·67	17·978273926	5·07	25·953918413
3·88	8·752590893	4·28	12·590316150	4·68	18·143659128	5·08	26·193837336
3·89	8·832321322	4·29	12·705569622	4·69	18·310586520	5·09	26·436002675
3·90	8·912787451	4·30	12·821892796	4·70	18·479070647	5·10	26·680435680
3·91	8·993996193	4·31	12·939295743	4·71	18·649126207		
3·92	9·075954517	4·32	13·057788626	4·72	18·820768025		
3·93	9·158669467	4·33	13·177381705	4·73	18·994011070		
3·94	9·242148147	4·34	13·298085340	4·74	19·168870460		
3·95	9·326397737	4·35	13·419909985	4·75	19·345361448		
3·96	9·411425473	4·36	13·542866196	4·76	19·523499439		
3·97	9·497238668	4·37	13·666964630	4·77	19·703299977		
3·98	9·583844704	4·38	13·792216043	4·78	19·884778763		
3·99	9·671251025	4·39	13·918631291	4·79	20·067951638		

x	$I_0(x)$	$I_1(x)$	$I_2(x)$
0·0	1·00000000000	Nil	Nil
0·2	1·01002502780	·100500834028	$\cdot 0^{2}501668751391$
0·4	1·04040178223	·204026755734	·0202680035615
0·6	1·09204536432	·313704025606	·0463652789678
0·8	1·16651492287	·432864802620	·0843529163180
1·0	1·26606587775	·565159103990	·135747669767
1·2	1·39372558413	·714677941552	·202595681546
1·4	1·55339509973	·886091981415	·287549411997
1·6	1·74998063974	1·08481063513	·393967345826
1·8	1·98955935662	1·31716723040	·526040211741
2·0	2·27958530233	1·59063685463	·688948447698
2·2	2·62914286357	1·91409465059	·889056817580
2·4	3·04925665799	2·29812381254	1·13415348087
2·6	3·55326890424	2·75538434051	1·43374248847
2·8	4·15729770350	3·30105582264	1·79940068733
3·0	4·88079258586	3·95337021738	2·24521244092
3·2	5·74720718718	4·73425389471	2·78829850299
3·4	6·78481316043	5·67010219264	3·44945892947
3·6	8·02768454705	6·79271460136	4·25395421296
3·8	9·51688802610	8·14042457894	5·23245403722
4·0	11·3019219521	9·75946515371	6·42218937528
4·2	13·4424561633	11·7056201430	7·86835133327
4·4	16·0104355250	14·0462213375	9·62578946244
4·6	19·0926234795	16·8625647618	11·7610735829
4·8	22·7936779931	20·2528346003	14·3549969097
5·0	27·2398718236	24·3356421424	17·5056149666
5·2	32·5835927106	29·2543098818	21·3319350638
5·4	39·0087877856	35·1820585061	25·9783957463
5·6	46·7375512926	42·3282880326	31·6203055668
5·8	56·0380968926	50·9461849787	38·4704468999
6·0	67·2344069764	61·3419367775	46·7870947172

x	$I_3(x)$	$I_4(x)$	$I_5(x)$
0·0	Nil	Nil	Nil
0·2	$\cdot 0^{3}167083750232$	$\cdot 0^{5}417500694777$	$\cdot 0^{7}834723214702$
0·4	$\cdot 0^{2}134672011869$	$\cdot 0^{4}672017811684$	$\cdot 0^{5}268449532285$
0·6	$\cdot 0^{2}460216582095$	$\cdot 0^{3}343620758320$	$\cdot 0^{4}205557100196$
0·8	·0111002210296	$\cdot 0^{2}110125859602$	$\cdot 0^{4}876350693866$
1·0	·0221684249243	$\cdot 0^{2}273712022104$	$\cdot 0^{3}271463155956$
1·2	·0393590030648	$\cdot 0^{2}580066622187$	$\cdot 0^{3}687894919051$
1·4	·0645222328531	·0110255569122	$\cdot 0^{2}151905049781$
1·6	·0998922705633	·0193713312135	$\cdot 0^{2}303561449592$
1·8	·148188982086	·0320769381221	$\cdot 0^{2}562481265409$
2·0	·212739959240	·0507285699791	$\cdot 0^{2}982567932312$
2·2	·297627709533	·0773448824914	·0163735913822
2·4	·407868011092	·114483453137	·0262565006355
2·6	·549626665935	·165373259392	·0407858678054
2·8	·730483412160	·234079089848	·0616860125932
3·0	·959753629490	·325705181936	·0912064776610
3·2	1·24888076598	·446647066782	·132263099020
3·4	1·61191521679	·604902664549	·188614829615
3·6	2·06609880918	·810456197666	·265085036586
3·8	2·63257822397	1·07575157832	·367838059088
4·0	3·33727577842	1·41627570765	·504724363113
4·2	4·21195220660	1·85127675241	·685710773430
4·4	5·29550364442	2·40464812914	·923416136884
4·6	6·63554425495	3·10601585905	1·23377754356
4·8	8·29033717554	3·99207544030	1·63687810838
5·0	10·3311501691	5·10823476364	2·15797454732
5·2	12·8451290635	6·51063229818	2·82877168171
5·4	15·9388023977	8·26861530445	3·68900194663
5·6	19·7423554848	10·4677818331	4·78838143757
5·8	24·4148422891	13·2137134973	6·18903056865
6·0	30·1505402994	16·6365544178	7·96846774238

x	$I_6(x)$	$I_7(x)$	$I_8(x)$
0·0	Nil	Nil	Nil
0·2	$\cdot0^{8}139087425642$	$\cdot0^{10}198660852119$	$\cdot0^{12}248291584037$
0·4	$\cdot0^{7}893980971214$	$\cdot0^{8}255240920874$	$\cdot0^{10}637748154995$
0·6	$\cdot0^{5}102559132723$	$\cdot0^{7}438834749717$	$\cdot0^{8}164357788982$
0·8	$\cdot0^{5}582022868887$	$\cdot0^{6}331639053615$	$\cdot0^{7}165452506106$
1·0	$\cdot0^{4}224886614771$	$\cdot0^{5}159921823120$	$\cdot0^{7}996062403333$
1·2	$\cdot0^{4}682085631142$	$\cdot0^{5}580928790861$	$\cdot0^{6}433537513798$
1·4	$\cdot0^{3}175196213558$	$\cdot0^{4}173686673046$	$\cdot0^{5}150954051219$
1·6	$\cdot0^{3}398740613950$	$\cdot0^{4}450598913012$	$\cdot0^{5}446656506452$
1·8	$\cdot0^{3}827978932673$	$\cdot0^{3}104953102941$	$\cdot0^{4}116770209099$
2·0	$\cdot0^{2}160017336352$	$\cdot0^{3}224639142001$	$\cdot0^{4}276993695123$
2·2	$\cdot0^{2}291946711786$	$\cdot0^{3}449225284743$	$\cdot0^{4}607607604085$
2·4	$\cdot0^{2}508136715570$	$\cdot0^{3}849664857007$	$\cdot0^{3}124988823159$
2·6	$\cdot0^{2}850453706344$	$\cdot0^{2}153415828186$	$\cdot0^{3}243684776533$
2·8	·0137719020155	$\cdot0^{2}266357538382$	$\cdot0^{3}454025096400$
3·0	·0216835897328	$\cdot0^{2}447211872992$	$\cdot0^{3}813702326455$
3·2	·0333248823452	$\cdot0^{2}729479022559$	$\cdot0^{2}141017510822$
3·4	·0501531656813	·0116036566222	$\cdot0^{2}237340311951$
3·6	·0741088738166	·0180554571973	$\cdot0^{2}389320693838$
3·8	·107756685981	·0275537875687	$\cdot0^{2}624273178058$
4·0	·154464799871	·0413299635012	$\cdot0^{2}980992761666$
4·2	·218632053769	·0610477626605	·0151395115677
4·4	·305975090770	·0889386166028	·0229885833970
4·6	·423890764347	·127975549614	·0343999611745
4·8	·581912714514	·182096322090	·0507984417519
5·0	·792285668997	·256488941728	·0741166321596
5·2	1·07068675643	·357956089960	·106958821921
5·4	1·43713021810	·495379239735	·152813670643
5·6	1·91710069457	·680308520630	·216329392995
5·8	2·54297113760	·927710973612	·303668787505
6·0	3·35577484714	1·25691804811	·422966068203

x	$I_9(x)$	$I_{10}(x)$	$I_{11}(x)$
0·0	Nil	Nil	Nil
0·2	$\cdot 0^{14}275848890728$	$\cdot 0^{16}275823817735$	$\cdot 0^{18}25072993174$
0·4	$\cdot 0^{11}141658875600$	$\cdot 0^{13}283214795193$	$\cdot 0^{15}514780037287$
0·6	$\cdot 0^{10}547312431307$	$\cdot 0^{11}164059590224$	$\cdot 0^{13}447130560011$
0·8	$\cdot 0^{9}734041402172$	$\cdot 0^{10}293190617555$	$\cdot 0^{11}106485828421$
1·0	$\cdot 0^{8}551838586274$	$\cdot 0^{9}275294803983$	$\cdot 0^{10}124897830849$
1·2	$\cdot 0^{7}287877246335$	$\cdot 0^{8}172164429560$	$\cdot 0^{10}9365304020$
1·4	$\cdot 0^{6}116775736690$	$\cdot 0^{8}813818331745$	$\cdot 0^{9}51597501248$
1·6	$\cdot 0^{6}394240656000$	$\cdot 0^{7}313576845153$	$\cdot 0^{8}22695995590$
1·8	$\cdot 0^{5}115736151949$	$\cdot 0^{6}103405714922$	$\cdot 0^{8}84091314720$
2·0	$\cdot 0^{5}304418590271$	$\cdot 0^{6}301696387935$	$\cdot 0^{7}272220233597$
2·2	$\cdot 0^{5}732884540826$	$\cdot 0^{6}797479795484$	$\cdot 0^{7}79029085679$
2·4	$\cdot 0^{4}164060359505$	$\cdot 0^{5}194355352977$	$\cdot 0^{6}20975653574$
2·6	$\cdot 0^{4}345596570386$	$\cdot 0^{5}442561241924$	$\cdot 0^{6}51648458289$
2·8	$\cdot 0^{4}691462615510$	$\cdot 0^{5}951341501153$	$\cdot 0^{5}11932971830$
3·0	$\cdot 0^{3}132372988831$	$\cdot 0^{4}194643934705$	$\cdot 0^{5}26103656940$
3·2	$\cdot 0^{3}243914684482$	$\cdot 0^{4}381550080109$	$\cdot 0^{5}544588441373$
3·4	$\cdot 0^{3}434700765661$	$\cdot 0^{4}720461248306$	$\cdot 0^{4}109000313633$
3·6	$\cdot 0^{3}752315248879$	$\cdot 0^{3}131630693989$	$\cdot 0^{4}210336156069$
3·8	$\cdot 0^{2}126860112417$	$\cdot 0^{3}233568560836$	$\cdot 0^{4}39292909243$
4·0	$\cdot 0^{2}209025303452$	$\cdot 0^{3}403788961327$	$\cdot 0^{4}713082278832$
4·2	$\cdot 0^{2}337343287863$	$\cdot 0^{3}681942087915$	$\cdot 0^{3}126089602839$
4·4	$\cdot 0^{2}534376788633$	$\cdot 0^{2}112771477116$	$\cdot 0^{3}217791653765$
4·6	$\cdot 0^{2}832351074598$	$\cdot 0^{2}182970173372$	$\cdot 0^{3}36828581676$
4·8	$\cdot 0127681829170$	$\cdot 0^{2}291775581322$	$\cdot 0^{3}61086702855$
5·0	$\cdot 0193157188168$	$\cdot 0^{2}458004441917$	$\cdot 0^{3}995541140110$
5·2	$\cdot 0288520225117$	$\cdot 0^{2}708643630312$	$\cdot 0^{2}159649826893$
5·4	$\cdot 0425979933861$	$\cdot 0108203593556$	$\cdot 0^{2}252258836536$
5·6	$\cdot 0622245406441$	$\cdot 0163219409248$	$\cdot 0^{2}393189448412$
5·8	$\cdot 0900039735967$	$\cdot 0243461108260$	$\cdot 0^{2}605186730033$
6·0	$\cdot 129008532906$	$\cdot 0359404694846$	$\cdot 0^{2}920696795753$

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